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An Introduction to Supersymmetry

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Abstract

Symmetries are the cornerstone of the modern development of theories in particle physics. The Standard Model, which describes the strong, weak and electromagnetic interactions is one of the most successful examples. Supersymmetry (SUSY) is a new formulation which is based on a symmetry that relates two basic types of elementary particles: bosons, which have integer spin, and fermions, which have half-integer spin. Since developed in the early 1970s, SUSY has drawn a growing attention, due to the interesting consequences that proposes. Despite some relevant phenomenological implications (as a suitable candidate for dark matter or the cancellations of quantum corrections for the Higgs boson), we will review some no-go theorems that leads us to consider SUSY as a suitable scenario in which spacetime and internal symmetries can be unified.

In this work we are going to study SUSY theories that contain particles with spin $s \leq 1$. To do so, we firstly investigate the main aspects of bosonic fields: scalar fields, Maxwell and Yang-Mills fields; and fermionic fields: Weyl, Dirac and Majorana spinors. The treatment of these fields has been done for any generic dimension. Finally, we have studied in detail two $\mathcal{N} = 1$ SUSY theories: we have considered the Wess-Zumino model, and the SUSY Yang-Mills theory. Explicit calculations and other aspects on group theory are provided in the various appendices.

In summary, we have learned the basics of SUSY theories, one of the most relevant developments in modern theoretical physics. To that end, we have studied the main properties of all the fields with spin $s \leq 1$ in full generality. We consider this work as a first step to address further open problems in theoretical physics.

Resumen

Las simetrías son la piedra angular en el desarrollo de teorías modernas de física de partículas. El Modelo Estándar, que describe las interacciones fuerte, débil y electromagnética, es uno de los ejemplos más exitosos. La supersimetría (SUSY) es una nueva formulación basada en una simetría que relaciona dos tipos básicos de partículas elementales: los bosones, que tienen espín entero, y los fermiones, que tienen espín semientero. Desde su desarrollo a principio de los años 1970, SUSY ha captado una creciente atención, debido a las interesantes consecuencias que propone. Pese a algunas de sus implicaciones fenomenológicas más relevantes (como una candidata adecuada para la materia oscura o las cancelaciones de las correcciones cuánticas al bosón de Higgs), revisaremos algunos teoremas de imposibilidad que llevan a considerar SUSY como un escenario apropiado en el que las simetrías internas y espaciotemporales pueden ser unificadas.

En este trabajo vamos a estudiar teorías SUSY que contienen partículas con espín $s \leq 1$. Para ello, investigamos primero los aspectos principales de los campos bosónicos: campos escalares, los campos de Maxwell y de Yang-Mills; y los campos fermiónicos: espinores de Weyl, Dirac y Majorana. El tratamiento de esos campos se ha hecho para dimensión genérica. Finalmente, hemos estudiado en detalle dos teorías SUSY con $\mathcal{N} = 1$: hemos considerado el modelo de Wess-Zumino, y la teoría SUSY Yang-Mills. Hemos provisto de cálculos explícitos y de otros aspectos de teoría de grupos en los diversos apéndices.

En síntesis, hemos aprendido las bases de la supersimetría, uno de los desarrollos más importantes en la física teórica moderna. Para este fin, hemos estudiado las principales propiedades de todos los campos de espín $s \leq 1$ con total generalidad. Consideramos este trabajo como un primer acercamiento para abordar otros problemas abiertos en la física teórica.

Contents

Introduction

Since many decades ago, particle physicists have tried to make sense out of the rich amount of data on elementary particles arising from high energy experiments [1]. To carry out this task, symmetry has been used. One of the most successful examples is the Standard Model (SM), which describes the electromagnetic, strong and weak interactions. This model is based on the symmetry group $SU(3) \times SU(2) \times U(1)$, that dictates and sculpts the SM Lagrangian.

But what is a symmetry? In few words, it is a transformation that leaves some system invariant. For example, a 90 degree rotation is a symmetry of the square. In a similar way, we say that some physical laws are invariant under certain symmetry transformations. For instance, Einstein's special theory of relativity shows that all physical laws have to be invariant under Lorentz transformations.

Despite the very recent discovery of the Higgs boson [2, 3], which plays an important role in the SM, there exist some physical phenomena in Nature that the SM does not explain: dark matter, the hierarchy problem, the matter-antimatter asymmetry, the description of gravity,.... This has led theorists to consider extensions of the SM. Formulations of this type are referred as *physics beyond the SM*. Some examples of these theories are grand unified theories, supersymmetry, brane-world scenarios, supergravities or superstrings, among others.

In this work we are going to study *supersymmetry (SUSY)*, which is a new symmetry that enlarges the type of symmetries of the SM. SUSY is a transformation that exchanges bosons by fermions and viceversa, thus stating that the physical laws are invariant under these transformations. This simple idea solves in an elegant manner some of the present enigmas of the SM all at once: it provides new particles as suitable candidates for dark matter, it implies the cancellation of the radiative corrections to the Higgs mass and it predicts the unification of the coupling constants at high energy scales [4]. Furthermore, SUSY seems to be a necessary ingredient to formulate a unification theory with gravity, as it brings the possibility of mixing internal and spacetime symmetries.

The goal of this work is to analyze in detail two SUSY theories: the Wess-Zumino model [5] and the supersymmetric Yang-Mills theory [6]. While the former involves spin-0 and spin-$\frac{1}{2}$ particles, the later contains spin-$\frac{1}{2}$ and spin-1 fields. To carry out this task, we previously need to study the most general aspects of bosonic and fermionic fields. For future research purposes, we study the conditions that dimensionality imposes on the fields when we consider arbitrary dimensions. In this work supersymmetry is considered at a classical level, since the construction of Lagrangians is essential for a quantum treatment. A further motivation is that it is precisely classical Lagrangians those which are required for the path integral formulation of quantum field theory.

Let us comment on the methodology of this work. We will study the main bibliographic references and explicitly reproduce the main results. Such calculations will be done analytically by hand and by using the Mathematica scientific software. We will make use of the open access repository arXiv and the programming language LaTeX for edition.

The thesis is organized as follows:

In Chapter 1 we present the Klein-Gordon scalar field, which describes bosons with spin $s = 0$. We discuss its equation of motion, as well as its internal and spacetime symmetries.

In Chapter 2 we explore the most significant features of spinors, which we apply to describe fermions with spin $s = \frac{1}{2}$. The discussion starts with Dirac spinors, and continues later with Majorana spinors, that can be regarded as Dirac spinors with a reality restriction. Majorana spinors are basic for the study of SUSY. Furthermore, we find that not all types of spinors exist in certain dimensions. This is a key aspect to formulate supersymmetric theories in dimensions higher than 4.

In Chapter 3 we analyze gauge fields describing bosonic particles with spin $s = 1$. We firstly study the Maxwell field, which enjoys an Abelian $U(1)$ gauge symmetry. Afterwards, we investigate Yang-Mills theory, which is a generalization of electromagnetism when other non-Abelian symmetry groups are considered. Yang-Mills theories constitute the basis for understanding the SM, since the groups $SU(3)$ and $SU(2)$ are non-Abelian.

In Chapter 4 we firstly ellaborate on further reasons to study supersymmetry. Then we discuss two important no-go theorems to introduce the superPoincaré algebra, which is an extension of the Poincaré algebra that mixes spacetime and internal symmetries in a non-trivial way. Finally we discuss two important $\mathcal{N} = 1$ SUSY theories that contain all types of fields studied in the previous chapters: the Wess-Zumino model and SUSY Yang-Mills.

Several appendices are provided. In Appendix A we present our notation. In Appendix B we develop the tools provided by Lagrangian and Canonical formalism. In Appendix C we review important concepts of group theory. In Appendix D we study basic notions of Clifford algebras and we apply them to study spinors in general dimension. Finally, in Appendix E we present some of the Mathematica code used.

Chapter 1

The Klein-Gordon scalar field

Scalar fields assign an scalar value to each point in space and time. For instance, the pressure distribution in a fluid is a scalar field. In this chapter we study the Klein-Gordon scalar field, owing his name to O. Klein and W. Gordon, who in 1926 used it to describe relativistic electrons. Although this scalar field exhibits Lorentz invariance, today we know it describes spin-0 particles, so it cannot account for the properties of the electron. The only elementary spin-0 particle known to date is the Higgs boson, in addition to other some non-elementary spin-0 particles in Nuclear Physics [7].

1.1 Equations of motion

We first consider the Klein-Gordon action for a set of real Klein-Gordon scalar fields $\phi^i(x)$ for $i = 1, ..., N$, defined on a D-dimensional Minkowski spacetime:

$$S = \int \mathrm{d}^D x \; \mathcal{L} = -\frac{1}{2} \int \mathrm{d}^D x \; [\eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i + m^2 \phi^i \phi^i]. \tag{1.1.1}$$

The equations of motion are obtained by $\dfrac{\delta S}{\delta \phi^i} = 0$. We proceed to derive them :

$$\frac{\partial \mathcal{L}}{\partial \phi^i} = -m^2 \phi^i, \tag{1.1.2}$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} = -\frac{\eta^{\alpha\beta}}{2} \left(\delta_{\alpha\mu} \partial_\beta \phi^i + \delta_{\beta\mu} \partial_\alpha \phi^i \right) = -\eta^{\mu\nu} \partial_\nu \phi^i, \tag{1.1.3}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} \right) = -\eta^{\mu\nu} \partial_\mu \partial_\nu \phi^i = -\Box \phi^i, \tag{1.1.4}$$

where $\Box \equiv \partial^\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_t^2 + \nabla_{D-1}^2$ is the d'Alembertian operator in D dimensions. Thus we arrive at

$$\boxed{(\Box - m^2) \phi^i = 0, \qquad i = 1, ...N.} \tag{1.1.5}$$

Each of the fields ϕ^i satisfies the equation of motion (1.1.5), commonly known as the Klein-Gordon equation. Because we have not considered any interaction interaction terms in our discussion, (1.1.5) is also referred as the free Klein-Gordon equation. It is instructive to look at its solutions [8].

The plane wave $e^{ip\cdot x} = e^{i(-Et+\vec{p}\cdot\vec{x})}$ constitutes a solution, as we obtain:

$$\Box e^{ip_\alpha x^\alpha} = \eta^{\mu\nu}\partial_\mu \left(ip_\nu e^{ip_\alpha x^\alpha}\right) = -\eta^{\mu\nu}p_\nu p_\mu e^{ip_\alpha x^\alpha} = m^2 e^{ip_\alpha x^\alpha}, \tag{1.1.6}$$

where we have made use of the relativistic dispersion relation $p^\mu p_\mu = \eta^{\mu\nu}p_\nu p_\mu = -E^2+\vec{p}^2 = -m^2$. Because of the linearity of the Klein-Gordon equation, any sum of solutions yields a new solution. We use this in order to write the general solution as a Fourier transform in the plane waves

$$\begin{aligned}\phi^i(\vec{x},t) &= \int \mathrm{d}E \int \frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{D-1}}\delta(E^2-\vec{p}^2-m^2)\tilde{\phi}^i(E,\vec{p})e^{i(-Et+\vec{p}\cdot\vec{x})} \\ &= \int \frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{D-1}2E}\left(a(\vec{p})e^{i(-Et+\vec{p}\cdot\vec{x})} + a^*(\vec{p})e^{i(Et+\vec{p}\cdot\vec{x})}\right).\end{aligned} \tag{1.1.7}$$

The factor $1/2E$ arises from the following property of the δ distribution

$$\delta(f(x)) = \sum_{x_0}\frac{\delta(x-x_0)}{|f'(x_0)|}, \quad \text{for all } x_0 \text{ such that } f(x_0)=0. \tag{1.1.8}$$

In our classical framework, the complex amplitudes $a(\vec{p})$ and $a^*(\vec{p})$ of the $(D-1)$ dimensional Fourier transform are simply functions of spatial momentum $\vec{p}$. In the quantized theory, they become anihilation and destruction operators for the particles described by the fields operators $\phi^i(x)$, which commute at different points of space $[\phi(\vec{x},0),\phi(\vec{y},0)]=0$.

1.2 Symmetries of the system

In this section, we are going to study different continuous symmetries associated with the Klein-Gordon field, as well as their corresponding Noether currents and charges.

1.2.1 Internal symmetries

We consider the mapping $\phi^i(x) \to \phi'^i(x) = R^i{}_j\phi^j(x)$, where $R^i{}_j$ is a $N\times N$ matrix of the special orthogonal group $SO(N)$ [1]. This global symmetry acts as a rotation on the internal space of the fields ϕ^i. This tranformation leaves the Klein-Gordon action invariant. For example, for the mass term we have

$$\phi'^i\phi'^i = R^i{}_jR^i{}_k\phi^j\phi^k = \delta_{jk}\phi^j\phi^k = \phi^j\phi^j. \tag{1.2.1}$$

[1]For a brief introduction on Lie groups, see Appendix C.2. A further discussion of the $SO(N)$ group is in Appendix C.2.2.

The same holds for $\partial_\mu \phi'^i$, as $R^i{}_j$ does not depend on x. From the theory of Lie algebras, we know that a matrix $R^i{}_j$ of the $SO(N)$ group can be given in terms of its generators $(t_A)^i{}_j$ by matrix exponentiation

$$R = e^{-\theta^A t_A}, \tag{1.2.2}$$

where θ^A for $A = 1, ..., N(N-1)/2$ are the independent parameters characterizing the transformation. This helps us to compute the corresponding Noether current using the general expression that can be found in Appendix (B.1.11). Identiying the parameters $\epsilon^A \to \theta^A$, we see that the infinitesimal variation is given by

$$\delta\phi^i \equiv (\delta^i{}_j - \theta^A (t_A)^i{}_j)\phi^j - \phi^i = -\theta^A (t_A)^i{}_j \phi^j = \epsilon^A \Delta_A \phi^i \longrightarrow \Delta_A \phi^i = -(t_A)^i{}_j \phi^j.$$

In addition,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = -\partial^\mu \phi^i. \tag{1.2.3}$$

As for this symmetry the Lagrangian density is invariant, we have that $K^\mu_A = 0$. The Noether currents are therefore

$$\boxed{J^\mu_A = -\partial_\mu \phi^i (t_A)^i{}_j \phi^j}. \tag{1.2.4}$$

And the conserved charges

$$Q_A = \int \mathrm{d}^{D-1}\vec{x}\; J^0{}_A = -\int \mathrm{d}^{D-1}\vec{x}\; \partial_0 \phi^i (t_A)^i{}_j \phi^j. \tag{1.2.5}$$

Let us consider a particular example, where $N = 2$. Thus, we can consider the $SO(2)$ group, whose unique generator is

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.2.6}$$

We explicitely check that this generator leads to a rotation 2×2 matrix upon exponentiation

$$R = e^{-\theta t} = \mathbb{1} - \theta t + \frac{\theta^2 t^2}{2} - \frac{\theta^3 t^3}{6} + O(\theta^4) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.2.7}$$

The transformation acts on the two fields as

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = R \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \cos\theta - \phi_2 \sin\theta \\ \phi_1 \sin\theta + \phi_2 \cos\theta \end{pmatrix}. \tag{1.2.8}$$

It is interesting to note that the same result can be obtained if we consider a single complex field given by $\phi = \phi_1 + i\phi_2$ and the following transformation

$$\phi' = e^{i\theta}\phi = (\cos\theta + i\sin\theta)(\phi_1 + i\phi_2) = \phi'_1 + i\phi'_2. \tag{1.2.9}$$

The complex number $e^{i\theta}$ is an element of the Unitary group $U(1)$. Now, inserting the generator t of (1.2.6) in the general expression (1.2.5) for the Noether charge, we get

$$Q = -\int \mathrm{d}^{D-1}\vec{x}\ (\partial_t\phi_2\phi_1 - \partial_t\phi_1\phi_2) = -\frac{1}{2}\int \mathrm{d}^{D-1}\vec{x}\ (\partial_t\phi\phi^* - \partial_t\phi^*\phi)\,. \qquad (1.2.10)$$

When the scalar fields are quantized, the quantity $\partial_t\phi_1\phi_2 - \partial_t\phi_2\phi_1$ is seen to be an electric charge density and so Q has the natural interpretation of an electric charge (for more details, see chapter 7 in [7]). In the case of a single real scalar field, $N = 1$, there is no internal space and so it cannot possess a conserved charge arising from any internal symmetry. That is why it is said that charged particles can only be described by complex fields.

1.2.2 Spacetime symmetries

Spacetime translations

The spacetime translation

$$\phi^i(x) \to \phi'^i(x) = \phi^i(x+a), \qquad (1.2.11)$$

for a constant vector a^μ, is a transformation that also leaves the Klein-Gordon action invariant, so it is a symmetry of the system. In order to obtain the Noether current we identify $\epsilon^A \to a^\nu$, with ν characterizing the transformation. In this case, $K^\mu{}_\nu \neq 0$ as

$$a^\nu\partial_\mu K^\mu{}_\nu = \delta\mathcal{L} = a^\nu\partial_\nu\mathcal{L} \quad\to\quad K^\mu{}_\nu = \delta^\mu{}_\nu\mathcal{L} \;\to\; K_{\mu\nu} = \eta_{\mu\nu}\mathcal{L}. \qquad (1.2.12)$$

The infinitesimal transformation $\Delta_A\phi^i$ now corresponds to $\partial_\nu\phi^i$. With this, we obtain that the Noether current is the so-called *Energy-momentum tensor* of the system

$$\boxed{J_{\mu_A} = \mathrm{T}_{\mu\nu} = \partial_\mu\phi^i\partial_\nu\phi^i + \eta_{\mu\nu}\mathcal{L}}. \qquad (1.2.13)$$

This tensor is important in physics, as it encompasses the density and the flux of both energy and momentum. For example, the element T_{00} represents energy density. The conserved charges are

$$\mathcal{P}_\mu = \int \mathrm{d}^{D-1}\vec{x}\ T_{0\mu}. \qquad (1.2.14)$$

It is worth noting that the charge

$$\begin{aligned}\mathcal{P}_0 &= \int \mathrm{d}^{D-1}\vec{x}\ T_{00} = \int \mathrm{d}^{D-1}\vec{x}\left[\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^i)}\partial_0\phi^i - \mathcal{L}\right]\\ &= \frac{1}{2}\int \mathrm{d}^{D-1}\vec{x}\left[(\partial_t\phi^i)^2 + |\vec{\nabla}_{D-1}\phi^i|^2 + (m\phi^i)^2\right] \end{aligned} \qquad (1.2.15)$$

is to be identified with the energy E of the system, which in this case is the same as the Hamiltonian H (we have given a definition of H in Appendix B.2). One can check that both the positive and negative frequency solutions appearing in (1.1.7) lead to a positive $\mathcal{P}_0$.

Lorentz transformations

We take a matrix Λ of the Lorentz group [2]. The transformation of a scalar field is given by:

$$\phi^i(x) \to \phi'^i(x) = \phi'^i(\Lambda x). \tag{1.2.16}$$

Here Λx is a shorthand notation for $\Lambda_\nu{}^\mu x^\nu$. The Klein-Gordon action is invariant under this transformation, and thus it is another symmetry of the system. We are going to prove this.

Proof. First note that the Klein-gordon Lagrangian density can also be expressed as $\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2)$ (we omit the index i because it does not play any role in the derivation). We are going to use that the derivative $\partial_\mu\phi(x)$ and the derivative $\partial^\mu\phi(x)$ follow the general transformation rules for covariant and contravariant vectors, respectively. Namely,

$$\partial^\mu\phi(x) \to (\Lambda^{-1})^\mu{}_\sigma(\partial^\sigma\phi)(\Lambda x), \qquad \partial_\mu\phi(x) \to (\Lambda^{-1})_\mu{}^\nu(\partial_\nu\phi)(\Lambda x) \tag{1.2.17}$$

These rules simply arise from the chain rule. For example, notice that $\frac{\partial}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial}{\partial x^\rho} = (\Lambda^{-1})_\nu{}^\rho\frac{\partial}{\partial x^\rho}$. We will also take into account the property $\Lambda^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu$. With all of this in mind, and calling $\overline{x}^\mu = \Lambda^\mu{}_\nu x^\nu$, we compute $S[\phi'(x)]$

$$\begin{aligned} S[\phi'(x)] &= -\frac{1}{2}\int \mathrm{d}^D x\left[(\Lambda^{-1})^\mu{}_\sigma(\Lambda^{-1})_\mu{}^\nu\partial^\sigma\phi(\overline{x})\partial_\nu\phi(\overline{x}) + m^2\phi^2(\overline{x})\right] \\ &= -\frac{1}{2}\int \mathrm{d}^D x\left[\Lambda_\sigma{}^\mu(\Lambda^{-1})_\mu{}^\nu\partial^\sigma\phi(\overline{x})\partial_\nu\phi(\overline{x}) + m^2\phi^2(\overline{x})\right] \\ &= -\frac{1}{2}\int \mathrm{d}^D \overline{x} J(\overline{x}, x)\left[\partial^\nu\phi(\overline{x})\partial_\nu\phi(\overline{x}) + m^2\phi^2(\overline{x})\right]. \end{aligned} \tag{1.2.18}$$

Now we need to compute the Jacobian. We will use that for an invertible matrix A we have $\det A^{-1} = (\det A)^{-1}$. We get:

$$J(\overline{x}, x) = \left|\det\left(\frac{\partial x^\mu}{\partial \overline{x}^\alpha}\right)\right| = \left|\det\left(\frac{\partial x^\alpha}{\partial \overline{x}^\mu}\right)\right|^{-1} = \left|\det\left(\Lambda^\alpha{}_\mu\right)\right|^{-1}. \tag{1.2.19}$$

But since we always consider proper Lorentz transformations, $\det\Lambda = 1$ and so $J(\overline{x}, x) = 1$. In conclusion, the action is invariant $S[\phi'(x)] = S[\phi(x)]$. □

The last step is to obtain the Noether current corresponding to this symmetry. We make the identification $\epsilon^A \to \lambda^{\rho\sigma}/2$, where $\lambda^{\rho\sigma}$ are antisymmetric numbers $\lambda^{\rho\sigma} = -\lambda^{\sigma\rho}$, denoting the $D(D-1)/2$ independent parameters of the Lorentz group. Note that

$$\delta\mathcal{L} = \frac{\lambda^{\rho\sigma}}{2}\partial_\mu K^\mu_{[\rho\sigma]} = \mathcal{L}(x^\mu + \lambda^{\mu\nu}x_\nu) - \mathcal{L}(x^\mu) = \partial_\rho\mathcal{L}\ \lambda^{\rho\sigma}x_\sigma. \tag{1.2.20}$$

[2] For a more detailed summary of the Lorentz group, see Appendix C.2.3.

Now, using the antisymmetry of $\lambda^{\rho\sigma}$, we can express the previous equation in the following way:

$$\frac{\lambda^{\rho\sigma}}{2}\partial_\mu K^\mu_{[\rho\sigma]} = \frac{\lambda^{\rho\sigma}}{2}\left(x_\sigma\partial_\rho\mathcal{L} - x_\rho\partial_\sigma\mathcal{L}\right). \tag{1.2.21}$$

This allows us to infer $K^\mu_{[\rho\sigma]}$:

$$K^\mu_{[\rho\sigma]} = x_\sigma\delta^\mu_\rho\mathcal{L} - x_\rho\delta^\mu_\sigma\mathcal{L}. \tag{1.2.22}$$

Finally, taking into account that the infinitesimal transformation corresponds in this case to $\Delta_A\phi^i = -\left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right)\phi^i$, we obtain the following Noether current:

$$J^\mu_A = \mathrm{M}^\mu{}_{[\rho\sigma]} = -x_\rho\partial^\mu\phi^i\partial_\sigma\phi^i + x_\sigma\partial^\mu\phi^i\partial_\rho\phi^i + x_\sigma\delta^\mu_\rho\mathcal{L} - x_\rho\delta^\mu_\sigma\mathcal{L}, \tag{1.2.23}$$

which can be rewritten as:

$$\boxed{\mathrm{M}^\mu{}_{[\rho\sigma]} = -x_\rho T^\mu{}_\sigma + x_\sigma T^\mu{}_\rho}. \tag{1.2.24}$$

$M^\mu{}_{[\rho\sigma]}$ is conserved, $\partial_\mu M^\mu{}_{[\rho\sigma]} = 0$, if $T_{\mu\nu}$ is both conserved, $\partial_\mu T^\mu{}_\nu = 0$, and symmetric $T_{\mu\nu} = T_{\nu\mu}$. The conserved charges are given by

$$\mathcal{M}_{[\rho\sigma]} \equiv \int \mathrm{d}^{D-1}\vec{x}\ M^0{}_{[\rho\sigma]}. \tag{1.2.25}$$

Chapter 2

Dirac and Majorana spinors

Spinors describe all the fermionic spin-$\frac{1}{2}$ particles existing in Nature, such as the electron and quarks. They were first introduced by the mathematician Élie Cartan in 1913 [9], but it was not until the 1920s that physicists started to use them to describe half-integer spin particles. In 1928 Dirac wrote his eponymous equation [10], considered as one of the greatest triumphs in physics. This equation assembled quantum mechanics and special relativity, explained the origin of the spin and predicted antimatter.

After Pauli had proposed neutrinos in 1930 to explain conservation of energy in beta decay experiments, it was suggested that neutrinos are their own antiparticles. During 1937 Majorana was pioneer in the study of such fermions [11].

2.1 Mathematical prelude

The Dirac equation requires a special representation of the Lorentz group, called spinor representation. The explicit description of spinor representation in dimension $D = 4$ is given through the homomorphism between the group of 2×2 complex matrices of unit modulus determinant, $SL(2, \mathbb{C})$, and the Lorentz group $SO^+(3, 1)$ (for the notation, see Appendix C.2.3). But before exploring this homomorphism between groups, it is convenient to see another important consequence of the case $D = 4$ at the level of the algebras: the study of the algebra of the Lorentz group, $\mathfrak{so}(3, 1)$, can be reduced to the study of the algebras of the $SU(2)$ group, $\mathfrak{su}(2)$. Let us investigate this powerful connection.

For $D = 4$, the Lorentz group contains six independent matrix generators $m_{[\mu\nu]}$, labelled with the antisymmetric indices $[\mu\nu]$. They consist of three spatial rotations $J_i \equiv -\frac{1}{2}\varepsilon_{ijk}m_{[jk]}$ and three boosts $K_i \equiv m_{[0i]}$ (for further details of these transformations, see Appendix C.2.3.1). The following generators I_k and I'_k

$$I_k = \frac{1}{2}(J_k - iK_k), \qquad I'_k = \frac{1}{2}(J_k + iK_k), \qquad k = 1, 2, 3, \tag{2.1.1}$$

satisfy the commutation relations of two independent copies of the Lie algebra $\mathfrak{su}(2)$:

$$\begin{aligned}[I_i, I_j] &= \varepsilon_{ijk} I_k, \\ [I'_i, I'_j] &= \varepsilon_{ijk} I'_k, \\ [I_i, I'_j] &= 0. \end{aligned} \tag{2.1.2}$$

We have included the proof for this at the end of Appendix C.2.3.1, due to its length. Because of (2.1.2), we see that the complexified Lie algebra of $\mathfrak{so}(3,1)$ is related to $\mathfrak{su}(2) \bigoplus \mathfrak{su}(2)$. The algebra of $SU(2)$ is well known from the theory of quantum angular momentum. This theory shows that the spin can be described by a basis $|jm\rangle$, where $j = 0, 1/2, 1, 3/2, ...$ and $m = -j, -j+1, ..., j-1, j$. Each j labels a different irreducible representation, and the number of m's gives the dimensionality of the representation.

Therefore, any finite and irreducible representation of $\mathfrak{so}(3,1)$ can be obtained as a product of two representations of $\mathfrak{su}(2)$ and classified by the pair of numbers (j, j'). The (j, j') representation has dimension $(2j+1)(2j'+1)$. This explains why a 4-dimensional representation of the generators of the Lorentz group is denoted by $(\frac{1}{2}, \frac{1}{2})$. This important result elucidates that the concept of spin is originated in Lorentz symmetry.

2.1.1 The homomorphism $SL(2,\mathbb{C}) \to SO^+(3,1)$

We study the 2 : 1 homomorphism [1] between the $SL(2,\mathbb{C})$ group and the Lorentz group $SO^+(3,1)$, which will allow us to obtain the generators of the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, the most basic spinor representations. The first remark is that an arbitrary 2×2 Hermitian matrix $\mathbf{x}$ can be parametrized as:

$$\mathbf{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}.$$

The second remark is that the determinant of $\mathbf{x}$ is minus the Minkowki norm,

$$\det \mathbf{x} = -(-x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3) = -x^\mu \eta_{\mu\nu} x^\nu. \tag{2.1.3}$$

Therefore, the vector space H of Hermitian 2×2 matrices and 4-dimensional Minkowski vector space M seem to have some type of connection. We proceed to show that there is an isomorphism [2] between them. For this task, we introduce two sets of 2×2 matrices:

$$\sigma_\mu = (-\mathbb{1}, \sigma_i), \qquad \overline{\sigma}_\mu = \sigma^\mu = (\mathbb{1}, \sigma_i), \tag{2.1.4}$$

[1] A homomorphism is a map between two algebraic structures of the same type that preserve the operation of the structures. When two different elements of an algebraic structure are mapped into a solely element of the other structure, we speak about a 2 : 1 homomorphism.

[2] An isomorphism is a bijective homomorphism, that is, it has an inverse. It is always 1 : 1. Do not confuse the isomorphism between H and M with the homomorphism between the groups of transformations acting on those spaces, namely $SL(2,\mathbb{C})$ and $SO^+(3,1)$, which we discuss here afterwards.

where $\mathbb{1}$ is the unit matrix, and the three Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.5}$$

The following identities hold:

$$\sigma_\mu \overline{\sigma}_\nu + \sigma_\nu \overline{\sigma}_\mu = 2\eta_{\mu\nu}\mathbb{1}, \tag{2.1.6}$$

$$\mathrm{tr}\left(\sigma^\mu \overline{\sigma}_\nu\right) = 2\delta^\mu{}_\nu. \tag{2.1.7}$$

Using (2.1.7), we find

$$\mathbf{x} = \mathbb{1}x^0 + \sigma_1 x^1 + \sigma_2 x^2 + \sigma_3 x^3 = \overline{\sigma}_\mu x^\mu, \tag{2.1.8}$$

$$\frac{1}{2}\mathrm{tr}\left(\sigma^\mu \mathbf{x}\right) = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu \overline{\sigma}_\nu x^\nu\right) = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu \overline{\sigma}_\nu\right) x^\nu = x^\mu. \tag{2.1.9}$$

This gives the explicit form of the isomorphism between spaces. We are almost ready for obtaining the desired homomorphism. Let A be a matrix of $SL(2,\mathbb{C})$, and consider the linear map:

$$\mathbf{x} \to \mathbf{x}' \equiv A\mathbf{x}A^\dagger. \tag{2.1.10}$$

The corresponding 4-vectors are related through

$$x'^\mu = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu \mathbf{x}'\right) = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu A\mathbf{x}A^\dagger\right) = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu A\overline{\sigma}_\nu A^\dagger\right) x^\nu \equiv \phi(A)^\mu{}_\nu x^\nu. \tag{2.1.11}$$

$\phi(A)^\mu{}_\nu$ is the homomorphism we were looking for. Transformation in (2.1.10) preserves the determinant, since $\det\mathbf{x}' = \det A \, \det\mathbf{x} \, \det A^\dagger = \det\mathbf{x}$. Therefore, the Minkowski norm is invariant under this transformation, and we can connect the homorphism with a transformation matrix Λ of the Lorentz group

$$\phi(A)^\mu{}_\nu = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu A\overline{\sigma}_\nu A^\dagger\right) = (\Lambda^{-1})^\mu{}_\nu. \tag{2.1.12}$$

Note that, for a given Λ^{-1}, there are two transformations corresponding to it, since there is a freedom in sign (as $\det A = \det(-A)$). Thus, $\phi(A) = \phi(-A) = \Lambda^{-1}$, and this is why we call this a $2:1$ homomorphism. Furthermore, we can convince ourselves that this is a homomorphism by showing that $\phi(A)\phi(B) = \phi(AB)$. For this purpose, let us consider the map $\mathbf{x}' = (AB)\mathbf{x}(AB)^\dagger = A\tilde{\mathbf{x}}A^\dagger$ where $\tilde{\mathbf{x}} \equiv B\mathbf{x}B^\dagger$. Then,

$$x'^\mu = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu A\overline{\sigma}_\nu A^\dagger\right)\tilde{x}^\nu = \frac{1}{2}\mathrm{tr}\left(\sigma^\mu A\overline{\sigma}_\nu A^\dagger\right)\frac{1}{2}\mathrm{tr}\left(\sigma^\nu B\overline{\sigma}_\rho B^\dagger\right) x^\rho \equiv \phi(A)^\mu{}_\nu \phi(B)^\nu{}_\rho x^\rho.$$

Two specially important relations are

$$A\overline{\sigma}_\mu A^\dagger = \overline{\sigma}_\nu (\Lambda^{-1})^\nu{}_\mu, \qquad A^\dagger \sigma_\mu A = \sigma_\nu \Lambda^\nu{}_\mu. \tag{2.1.13}$$

They can be proven without many difficulties. For example, $\mathbf{x}'$ can be expressed in two ways:

$$\mathbf{x}' = A\mathbf{x}A^\dagger = A\overline{\sigma}_\mu A^\dagger x^\mu,$$
$$\mathbf{x}' = \overline{\sigma}_\nu x'^\nu = \overline{\sigma}_\nu (\Lambda^{-1})^\nu{}_\mu x^\mu.$$

Then we straightforwardly read off the first identity. Equations (2.1.13) provide the recipe for obtaining a Lorentz transformation Λ from a matrix A of the $SL(2,\mathbb{C})$ group.

For the next step of the discussion, we present two sets of matrices, given in terms of $\overline{\sigma}_\nu$ and σ_μ as:

$$\sigma_{\mu\nu} = \frac{1}{4}\left(\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu\right), \tag{2.1.14}$$
$$\overline{\sigma}_{\mu\nu} = \frac{1}{4}\left(\overline{\sigma}_\mu\sigma_\nu - \overline{\sigma}_\nu\sigma_\mu\right). \tag{2.1.15}$$

The commutator algebras of these matrices are the same as those of the Lorentz group. For example, using (2.1.6) one can show

$$\left[\sigma_{[\mu\nu]}, \sigma_{[\rho\sigma]}\right] = \eta_{\nu\rho}\sigma_{[\mu\sigma]} - \eta_{\mu\rho}\sigma_{[\nu\sigma]} - \eta_{\nu\sigma}\sigma_{[\mu\rho]} + \eta_{\mu\sigma}\sigma_{[\nu\rho]}. \tag{2.1.16}$$

According to (2.1.1), the commutators of $I_k = -\frac{1}{2}(\frac{1}{2}\varepsilon_{ijk}\sigma_{ij} + i\sigma_{0k})$ and $I'_k = -\frac{1}{2}(\frac{1}{2}\varepsilon_{ijk}\sigma_{ij} - i\sigma_{0k})$ should satisfy (2.1.2). This is the case if $I_k = 0$. We thus arrive at the important conclusion that the matrices $\sigma_{\mu\nu}$ and $\overline{\sigma}_{\mu\nu}$ are generators in the $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ representations. Their exponentiation gives a representation of the Lorentz group, but acting in the space H of Hermitian matrices instead of the Minkowski space M. This is precisely the mapping involving A in (2.1.10), so we can identify

$$A = e^{-\frac{1}{2}\lambda^{\mu\nu}\sigma_{\mu\nu}}, \tag{2.1.17}$$
$$A^\dagger = e^{\frac{1}{2}\lambda^{\mu\nu}\overline{\sigma}_{\mu\nu}}, \tag{2.1.18}$$

where $\lambda^{\mu\nu}$ are the parameters of the transformations. Notice that this identification is consistent since $\sigma^\dagger_{\mu\nu} = -\overline{\sigma}_{\mu\nu}$.

2.1.2 Spinors are not vectors

Spinors, which we may call as ψ^α, are two-component complex objects. That is, they live in $\mathbb{C}^2$. What is characteristic about them is their transformation properties. Spinors transforms under the $SL(2,\mathbb{C})$ group which, as we have just seen, is homomorphic to the Lorentz group. This is called *the spinor representation*. Strictly speaking, the $SL(2,\mathbb{C})$ group is the universal covering group of the Lorentz group (for a formal definition of universal covering group, see [12]).

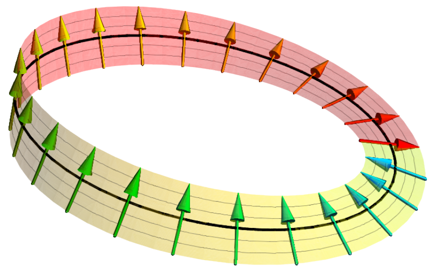

Figure 2.1: A spinor visualized as an arrow pointing along the Möbius strip. Picture taken from [13].

Thus, as spinors transform according to the $SL(2,\mathbb{C})$ group and not the Lorentz group, it should be not surprising that they don't behave as normal vectors. So as to exemplify this, let us consider a rotation of 2π around the 3-axis. We take (2.1.17) for an antisymmetric parameter $\lambda^{21} = -\lambda^{12} = \varphi$. Thus,

$$A = e^{-\frac{1}{2}(\lambda^{21}\sigma_{21}+\lambda^{12}\sigma_{12})} = e^{-\frac{\varphi}{2}(\sigma_{21}-\sigma_{12})} = e^{-\frac{\varphi}{4}(\sigma_2\sigma_1-\sigma_1\sigma_2)} = e^{i\frac{\varphi}{2}\sigma_3}, \tag{2.1.19}$$

where in the last step we have used the property $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. Remembering the general formula for the matrix exponential of Pauli matrices, $e^{ia\sigma_j} = \mathbb{1}\cos a + i\sigma_j \sin a$, we see that for a rotation of $\varphi = 2\pi$ we get

$$A = \mathbb{1}\cos\pi + i\sigma_3 \sin\pi = -\mathbb{1}. \tag{2.1.20}$$

This means that under a rotation of 360º a spinor reverses its direction, $\psi^\alpha \to -\psi^\alpha$, which it is definitely not what happens to a vector! In fact, a spinor needs a rotation of 720º in order to return to its original position. We can get an intuitive picture of this if we imagine the spinor as an arrow sliding across a Möbius strip (see Figure 2.1).

2.2 Dirac spinors

Here we follow a different approach from what we did for the Klein-Gordon scalar field. We first discuss the equation of motion and its implications, and later we construct a suitable action for the theory.

2.2.1 The Dirac equation

Dirac postulated that a free electron is described by the following equation of motion [3]

$$\boxed{\partial\!\!\!/\Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x)}. \tag{2.2.1}$$

[3] This is the classical Dirac equation. The quantum version of this equation includes a factor i in front of the derivatives because of the pressence of the hermitian momentum operator.

Here $\Psi(x)$ is a complex multicomponent field that transforms under some representation of the Lorentz group. It is closely related to the basic spinor representations we have discussed in the previous section. In fact, for $D = 4$, we are going to see that Ψ is formed by two spinors, and it is normally called bispinor or Dirac spinor. The quantities γ^μ, $\mu = 0, 1, ..., D-1$, are a set of square matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}. \tag{2.2.2}$$

The Dirac equation mixes up different components of Ψ through the matrices γ^μ, but each individual component itself solves the Klein-Gordon equation. To see this, we write

$$(\gamma^\nu\partial_\nu + m)(\gamma^\mu\partial_\mu - m)\Psi = (\gamma^\mu\gamma^\nu\partial_\nu\partial_\mu - m^2)\Psi = 0. \tag{2.2.3}$$

But because of (2.2.2) we see $\gamma^\mu\gamma^\nu\partial_\nu\partial_\mu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\nu\partial_\mu = \partial^\mu\partial_\mu$, so (2.2.3) becomes the Klein-Gordon equation for each component Ψ^α.

Although we are not going to enter into explanatory details, it is worth saying that after canonical quantization, the fields $\Psi(x)$ become operators that anticommute at different points in space, as opposed to the scalar field $\phi(x)$, which becomes a commuting operator. This is a manifestation of the spin statistics theorem [4]. But even in the classical case, the components of Ψ are required to be anti-commuting Grassmann numbers [5], satisfying

$$\{\Psi_\alpha(x), \Psi_\beta(y)\} = 0. \tag{2.2.4}$$

We will understand this requirement later, when studying Majorana spinors.

The condition (2.2.2) is the defining condition for the generators of a Clifford algebra. The structure of this algebra is discussed in Appendix D. We now write a well-known representation of the γ-matrices for $D = 4$, called Weyl representation, in which the 4×4 γ^μ have the 2×2 matrices of (2.1.4) in off-diagonal blocks:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.2.5}$$

There are block off-diagonal representations of this type in all even dimensions, as we show in Appendix D.1.2. We are going to prove that the following commutators

$$\Sigma^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu, \gamma^\nu] \tag{2.2.6}$$

satisfy the commutation relations (2.1.16) and, as a consequence, they form also a representation of the Lie algebra of the Lorentz group.

[4] This theorem states that multiparticle states described by fermions and bosons need to be antisymmetric and symmetric under interchange of two particles, respectively. It is then said that fermions obey Fermi-Dirac statistics whereas bosons obey Bose-Einstein statistics. For a detailed discussion, see [14]

[5] Grassmann numbers θ^i are real numbers that belong to an algebra in which all elements anticommute between them, $\theta_i\theta_j = -\theta_j\theta_i$.

***Proof*.** First we need to show $[\Sigma^{\mu\nu}, \gamma^\rho] = \gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\mu\rho}$. Note we can write the following $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \mathbb{1}\eta^{\mu\nu} + 2\Sigma^{\mu\nu}$. Since the identity $\mathbb{1}$ commutes with everything, for any matrix X we have:

$$[X, \Sigma^{\mu\nu}] = \frac{1}{2}[X, \gamma^\mu\gamma^\nu - \mathbb{1}\eta^{\mu\nu}] = \frac{1}{2}[X, \gamma^\mu\gamma^\nu]. \tag{2.2.7}$$

If we now use the matrix identity $[A, BC] = \{A, B\}C - B\{A, C\}$ and (2.2.2), we have:

$$[\gamma^\rho, \gamma^\mu\gamma^\nu] = \{\gamma^\rho, \gamma^\mu\}\gamma^\nu - \gamma^\mu\{\gamma^\rho, \gamma^\nu\} = 2\eta^{\rho\mu}\gamma^\nu - 2\eta^{\rho\nu}\gamma^\mu, \tag{2.2.8}$$

Then $[\Sigma^{\mu\nu}, \gamma^\rho] = -\frac{1}{2}[\gamma^\rho, \gamma^\mu\gamma^\nu] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}$. Now we compute the term $[\gamma^\rho\gamma^\sigma, \Sigma^{\mu\nu}]$:

$$\begin{aligned}
[\gamma^\rho\gamma^\sigma, \Sigma^{\mu\nu}] &= \gamma^\rho[\gamma^\rho, \Sigma^{\mu\nu}] + [\gamma^\rho, \Sigma^{\mu\nu}]\gamma^\sigma \\
&= \gamma^\rho\left(\eta^{\sigma\mu}\gamma^\nu - \eta^{\sigma\nu}\gamma^\mu\right) + \left(\eta^{\rho\mu}\gamma^\nu - \eta^{\rho\nu}\gamma^\mu\right)\gamma^\sigma \\
&= \eta^{\sigma\mu}\gamma^\rho\gamma^\nu - \eta^{\sigma\nu}\gamma^\rho\gamma^\mu + \eta^{\rho\mu}\gamma^\nu\gamma^\sigma - \eta^{\rho\nu}\gamma^\mu\gamma^\sigma \\
&= 2\eta_{\sigma\mu}\Sigma^{\rho\nu} - 2\eta_{\sigma\nu}\Sigma^{\rho\mu} + 2\eta_{\rho\mu}\Sigma^{\nu\sigma} - 2\eta_{\rho\nu}\Sigma^{\mu\sigma}.
\end{aligned}$$

And in conclusion, using the symmetry of $\eta_{\mu\nu}$ and the antisymmetry of $\Sigma^{\mu\nu}$, we have

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = -\frac{1}{2}[\gamma^\rho\gamma^\sigma, \Sigma^{\mu\nu}] = \eta_{\nu\rho}\Sigma^{\mu\sigma} - \eta_{\mu\rho}\Sigma^{\nu\sigma} - \eta_{\sigma\nu}\Sigma^{\mu\rho} + \eta_{\mu\sigma}\Sigma^{\nu\rho}. \tag{2.2.9}$$

□

In the Weyl representation, one sees that the matrices $\Sigma_{\mu\nu}$ are expressed in terms of the 2×2 matrices $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ as

$$\Sigma_{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}. \tag{2.2.10}$$

From (2.2.10), we see that the 4-dimensional representation of $\mathfrak{so}(3,1)$ given by $\Sigma_{\mu\nu}$ is block-diagonal and therefore reducible. Actually, it is a direct sum of the irreducible $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations given by $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ that we discussed in the previous section. Lorentz transformations on Dirac spinors are implemented as

$$L = e^{\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}} \tag{2.2.11}$$

In Appendix D.1 we give an explicit construction of the γ-matrices, which shows they are necessarily complex. Moreover, the matrix γ^0 is Hermitian while the rest of matrices γ^i are anti-Hermitian. This explains why the Dirac field needs to be complex. In other words, if it was chosen to be real, then any arbitrary Lorentz transformation would transform it into a complex one.

We check now that the Dirac equation is Lorentz covariant, as it should be. This means that, if $\Psi(x)$ is a solution, then $\Psi'(x) = L^{-1}\Psi(\Lambda x)$ is also a solution.

***Proof*.** In the first place, we have to prove $L\gamma^\rho L^{-1} = \gamma^\sigma \Lambda_\sigma{}^\rho$. We will use the Hadamard Lemma, which states that for any 2 matrices A and B

$$e^{-B} A e^{B} = A + [A, B] + \frac{1}{2}[[A, B], B] + ... \tag{2.2.12}$$

So, by choosing $A = \gamma^\rho$ and $B = -\frac{1}{2}\lambda_{\mu\nu}\Sigma^{\mu\nu}$, and taking into account that $[\gamma^\rho, B] = -\frac{\lambda_{\mu\nu}}{2}[\gamma^\rho, \Sigma^{\mu\nu}] = \frac{\lambda_{\mu\nu}}{2}(\gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}) = \lambda_\mu{}^\rho\gamma^\mu$, we have:

$$\begin{aligned} L(\lambda)\gamma^\rho L(\lambda)^{-1} &= \gamma^\rho + [\gamma^\rho, B] + \frac{1}{2}[[\gamma^\rho, B], B] + ... = \gamma^\rho + \lambda_\mu{}^\rho\gamma^\mu + \frac{1}{2}\lambda_\alpha{}^\rho\lambda_\nu{}^\alpha\gamma^\nu + ... \\ &= \gamma^\sigma\left(\delta_\sigma^\rho + \lambda_\sigma{}^\rho + \frac{1}{2}\lambda_\alpha{}^\rho\lambda_\sigma{}^\alpha ...\right) = \gamma^\sigma\Lambda_\sigma{}^\rho. \end{aligned} \tag{2.2.13}$$

Now we compute $\left(\gamma^\mu\partial'_\mu - m\right)\Psi'(x)$ and we make use of the previous relation in order to check that this is zero

$$\begin{aligned} &\left(\gamma^\mu\frac{\partial}{\partial x'^\mu} - m\right)\Psi'(x) = \left(\gamma^\mu\frac{\partial}{\partial x'^\mu} - m\right)L^{-1}\Psi(x') = \left(\gamma^\mu\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} - m\right)L^{-1}\Psi(x') \\ &= \left(\gamma^\mu(\Lambda^{-1})_\mu{}^\nu\frac{\partial}{\partial x^\nu} - m\right)L^{-1}\Psi(x') = [(\Lambda^{-1})_\mu{}^\nu\gamma^\mu L^{-1}]\partial_\nu\Psi(x) - L^{-1}m\Psi(x'). \end{aligned}$$

If we multiply by L^{-1} each side of (2.2.13), we get $\gamma^\mu L^{-1} = L^{-1}\gamma^\sigma\Lambda_\sigma{}^\mu$. Introducing this relation above we see

$$\begin{aligned} \left(\gamma^\mu\partial'_\mu - m\right)\Psi'(x) &= [\underbrace{\Lambda_\sigma{}^\mu(\Lambda^{-1})_\mu{}^\nu}_{\delta_\sigma^\nu} L^{-1}\gamma^\sigma]\partial_\nu\Psi(x') - L^{-1}m\Psi(x') \\ &= L^{-1}\left(\gamma^\nu\partial_\nu - m\right)\Psi(x') = 0. \end{aligned} \tag{2.2.14}$$

□

2.2.2 Constructing the Dirac action

We need to build a suitable Lorentz invariant action. For this purpose, we have to introduce a bilinear form that satisfies Lorentz invariance. This is some scalar quantity, formed by the product $\Psi^\dagger\beta\Psi$, with β a square matrix to be found. Under an infinitesimal Lorentz transformation, the variations of Ψ and $\Psi^\dagger$ are:

$$\begin{aligned} \delta\Psi(x) &= -\frac{1}{2}\lambda^{\mu\nu}\left(\Sigma_{\mu\nu} + L_{[\mu\nu]}\right)\Psi(x) = -\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi(x) + \lambda^\mu{}_\nu x^\nu\partial_\mu\Psi(x), \\ \delta\Psi^\dagger(x) &= -\frac{1}{2}\lambda^{\mu\nu}\Psi^\dagger\Sigma^\dagger_{\mu\nu} + \lambda^\mu{}_\nu x^\nu\partial_\mu\Psi(x)^\dagger. \end{aligned} \tag{2.2.15}$$

Lorentz invariance requires

$$\begin{aligned} \delta(\Psi^\dagger\beta\Psi) &= \lambda^\mu{}_\nu x^\nu\partial_\mu(\Psi^\dagger\beta\Psi) = \delta\Psi^\dagger(\beta\Psi) + (\Psi^\dagger\beta)\delta\Psi \\ &= -\frac{1}{2}\lambda^{\mu\nu}\Psi^\dagger\left(\Sigma^\dagger_{\mu\nu}\beta + \beta\Sigma_{\mu\nu}\right)\Psi + \lambda^\mu{}_\nu x^\nu\partial_\mu\left(\Psi^\dagger\beta\Psi\right). \end{aligned} \tag{2.2.16}$$

Therefore the following condition needs to be fulfilled:

$$\Sigma_{\mu\nu}^{\dagger}\beta + \beta\Sigma_{\mu\nu} = 0. \tag{2.2.17}$$

We look for a real bilinear form, so we choose a Hermitian matrix $\beta = \beta^{\dagger}$. If the Lorentz group were compact, it would have finite-dimensional unitary representations, which would imply that the generators of its Lie algebra are all anti-Hermitian [15]. Then in (2.2.17) it would be enough to choose β as the identity, so that $\Psi^{\dagger}\Psi$ would be the Lorentz scalar. The problem is that the Lorentz group is non-compact and therefore it has no finite-dimensional unitary representations. The required anti-Hermitian property holds for spatial rotations

$$\Sigma_{ij} = \frac{1}{4}[\gamma^i, \gamma^j] = \frac{1}{4}\begin{pmatrix} \sigma^i\sigma^j - \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j - \sigma^i\sigma^j \end{pmatrix} = i\frac{\epsilon^{ijk}}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \to \Sigma_{ij}^{\dagger} = -\Sigma_{ij}$$

but not for boosts

$$\Sigma_{0i} = \frac{1}{4}[\gamma^0, \gamma^i] = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \to \Sigma_{0i}^{\dagger} = \Sigma_{0i}$$

which are Hermitian. Therefore β cannot be the identity. An alternative is to take β to be any multiple of γ^0, since then (2.2.17) is satisfied. We check this. First we compute

$$\gamma^0\gamma^\mu(\gamma^0)^{-1} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\bar{\sigma}^\mu \\ -\sigma^\mu & 0 \end{pmatrix} = -(\gamma^\mu)^{\dagger}, \tag{2.2.18}$$

which in turn implies

$$\gamma^0\Sigma_{\mu\nu}(\gamma^0)^{-1} = \frac{1}{4}\left((\gamma_\mu)^{\dagger}(\gamma_\nu)^{\dagger} - (\gamma_\nu)^{\dagger}(\gamma_\mu)^{\dagger}\right) = \frac{1}{4}\left(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu\right)^{\dagger} = -\Sigma_{\mu\nu}^{\dagger}.$$

It is convenient to choose $\beta = i\gamma^0$. With this, we can define the Dirac adjoint (a row vector) by

$$\bar{\Psi} \equiv \Psi^{\dagger}\beta = \Psi^{\dagger}i\gamma^0, \tag{2.2.19}$$

so that we can write the invariant bilinear form as $\bar{\Psi}\Psi$. We have everything we need to define the action of the free Dirac field:

$$S[\bar{\Psi}, \Psi] = \int \mathrm{d}^D x\, \mathcal{L} = \int \mathrm{d}^D x\, [-\bar{\Psi}\gamma^\mu\partial_\mu\Psi + m\bar{\Psi}\Psi]. \tag{2.2.20}$$

Integrating by parts and setting to zero the term with a total derivative, this action can equivalently be written as

$$S[\bar{\Psi}, \Psi] = \int \mathrm{d}^D x\, [\partial_\mu\bar{\Psi}\gamma^\mu\Psi + m\bar{\Psi}\Psi]. \tag{2.2.21}$$

The condition that the action is stationary, $\delta S[\bar{\Psi}, \Psi] = 0$, leads, by using (2.2.21) and (2.2.20), to the two equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)} \right) = 0 \quad \rightarrow \quad \partial_\mu \bar{\Psi} \gamma^\mu - m\bar{\Psi} = 0, \tag{2.2.22}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \bar{\Psi}\right)} \right) = 0 \quad \rightarrow \quad [\gamma^\mu \partial_\mu - m]\Psi = 0. \tag{2.2.23}$$

One of them is the already discussed Dirac equation. The other one is its conjugate.

2.2.3 Left or right?

The representation we saw in (2.2.10) for $D = 4$ is reducible, so we can always write

$$\Psi = \begin{pmatrix} \boldsymbol{\chi} \\ \boldsymbol{\eta} \end{pmatrix}. \tag{2.2.24}$$

Here $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$ are spinors that transform according to (2.1.17) and (2.1.18), whereas the Dirac spinor Ψ transform according to (2.2.11). Actually this can be done for any even dimension $D = 2m$, since off-diagonal block representations for γ^μ exist for all even dimensions. Spinors $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$, that transform under irreducible representations, are more fundamental objects than the Dirac spinor, and they are tipically called *Weyl spinors* (we will normally writen them in bold). In even general dimension, γ^μ are $2^m \times 2^m$ matrices (as we show in Appendix D.1), so Dirac spinors have to have 2^m components and Weyl spinors $2^{(m-1)}$ components.

Weyl spinors have a definite **chirality**. Chirality is described by the eigenvalues of the chiral matrix (for the details, see Appendix D.1.2). In $D = 4$ the chiral matrix is the so-called γ_5 and in the Weyl representation is given by

$$\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.2.25}$$

Notice that setting either $\boldsymbol{\chi}$ or $\boldsymbol{\eta}$ to zero in (2.2.24) yields eigenstates of γ_5:

$$\gamma_5 \begin{pmatrix} \boldsymbol{\chi} \\ 0 \end{pmatrix} = + \begin{pmatrix} \boldsymbol{\chi} \\ 0 \end{pmatrix}, \qquad \gamma_5 \begin{pmatrix} 0 \\ \boldsymbol{\eta} \end{pmatrix} = - \begin{pmatrix} 0 \\ \boldsymbol{\eta} \end{pmatrix}. \tag{2.2.26}$$

Particles with positive chirality, such as $\boldsymbol{\chi}$, are said to be *left-chiral*, whereas particles with negative chirality, such as $\boldsymbol{\eta}$, are said to be *right-chiral*. That is why sometimes we find Ψ_L and Ψ_R as an alternative notation for $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$. As we show in D.1.2, one can define the matrices $P_{L,R} \equiv \frac{1}{2}(1 \pm \gamma_5)$, which project a Dirac spinor onto some of the two representations. For example, $\Psi_L = P_L \Psi$. As we see, *chirality tells us under which representation of the Lorentz group a spinor is transformed*. It can be shown that left-chiral and right-chiral particles are related through a parity transformation.

Let us express Dirac equation (2.2.1) in terms of the Weyl fields. In the Weyl representation (2.2.5) for the γ-matrices, Dirac equation splits up in the two equations

$$\bar{\sigma}^{\mu}\partial_{\mu}\boldsymbol{\chi}(x) = m\boldsymbol{\eta}(x), \qquad \sigma^{\mu}\partial_{\mu}\boldsymbol{\eta}(x) = m\boldsymbol{\chi}(x). \tag{2.2.27}$$

We see that Dirac equation in general couples the Weyl spinors $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$. Interestingly, in the case of zero mass $m = 0$, equations are decoupled:

$$\bar{\sigma}^{\mu}\partial_{\mu}\boldsymbol{\chi}(x) = 0, \qquad \sigma^{\mu}\partial_{\mu}\boldsymbol{\eta}(x) = 0. \tag{2.2.28}$$

These are called *Weyl equations*. As these equations are independent, we can now think of $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$ as describing two different particles, instead of two states of a single particle. In fact, there are theories that can contain only left-chiral or right-chiral particles. This is the case of the Standard Model, which contains only left-chiral neutrinos [6].

It is the moment to introduce helicity. **Helicity** is defined as the projection of the spin along the direction of motion of the particle. If we take the spin operator $\vec{S} = \frac{1}{2}\vec{\sigma}$ and the momentum operator $\vec{p} = (\partial_1, \partial_2, \partial_3)$, we can define the helicity operator h as

$$h \equiv \vec{S} \cdot \hat{p} = \vec{S} \cdot \frac{\vec{p}}{|\vec{p}|}. \tag{2.2.29}$$

Now, equations (2.2.28) can be reexpressed in terms of the 4-momentum operator $P_{\mu} = \partial_{\mu}$, or in components $P = (p_0, \vec{p})$. Using that for massless particles $p_0 = |\vec{p}|$, we can write Weyl equations as

$$h\boldsymbol{\chi} = +\frac{1}{2}\boldsymbol{\chi}, \qquad h\boldsymbol{\eta} = -\frac{1}{2}\boldsymbol{\eta}. \tag{2.2.30}$$

Thus, in the massless limit, Weyl spinors are eigenstates of helicity. Particles with $+\frac{1}{2}$ helicity eigenvalue are said to be *left-handed*, as opposed to particles with $-\frac{1}{2}$ helicity eigenvalue, called *right-handed*. As we see, helicity and chirality are equivalent concepts only for the massless case. In general, for massive particles, the right-chiral and left-chiral spinors $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$ will be linear combinations eigenstates of helicity.

We can understand this from a physical point of view: for a massive particle, it is always possible to boost to a reference frame where the direction of motion is seen reversed. Therefore, an observer can see a left-handed particle but other may observe that the particle is right-handed. For massless particles, this is not possible since they travel at the speed of light [17].

[6] Only left-chiral neutrinos are allowed because parity is violated in weak interactions. Until two decades ago, the Standard Model considered neutrinos to be massless. However, neutrino oscillation experiments have showed that neutrinos actually have mass (see [16]). Thus they cannot be Weyl fields. It has been suggested that they may be Majorana particles because of their neutral charge, but the experimental situation remains inconclusive.

2.2.3.1 Solution of the Dirac equation for $D = 4$

As we have explained, each component of the Dirac spinor solves the Klein-Gordon equation independently, so the Dirac equation also accepts plane-wave solutions. We write positive and negative frequency solutions as $\boldsymbol{u}(\vec{p}, s)e^{i(\vec{p}\cdot\vec{x}-Et)}$ and $\boldsymbol{v}(\vec{p}, s)e^{-i(\vec{p}\cdot\vec{x}-Et)}$, respectively. The momentum space functions $\boldsymbol{u}(\vec{p}, s)$ and $\boldsymbol{v}(\vec{p}, s)$ denote independent column vectors, with the same number of components as Ψ. The new feature is the discrete label s. Why do we need to introduce it? What are their values? Let us pause and discuss the degrees of freedom of the Dirac spinor first [7].

The number of degrees of freedom of a system is half the dimension of the phase space. As a consequence of the fact that the Dirac equation is of first order, the momentum is not proportional to the derivative of Ψ. In fact, using (2.2.20)

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \Psi)} = i\Psi^\dagger \tag{2.2.31}$$

Thus, the configuration space has 8 real dimensions (since Ψ has four complex components) and we can conclude that the number of degrees of freedom is 4. Two of them are given by positive and negative frequency solutions. The other two are encoded in the label s, so s run over two values [8]. Exploiting the linearity of the Dirac equation, a general solution can thus be expressed as the Fourier expansion

$$\begin{aligned}
\Psi(x) &= \Psi_+(x) + \Psi_-(x), \quad \text{with} \\
\Psi_+(x) &= \int \frac{\mathrm{d}^3\vec{p}}{2E(2\pi)^3} e^{i(\vec{p}\cdot\vec{x}-Et)} \sum_{s=1,2} c(\vec{p}, s)\boldsymbol{u}(\vec{p}, s), \\
\Psi_-(x) &= \int \frac{\mathrm{d}^3\vec{p}}{2E(2\pi)^3} e^{-i(\vec{p}\cdot\vec{x}-Et)} \sum_{s=1,2} d(\vec{p}, s)^*\boldsymbol{v}(\vec{p}, s).
\end{aligned} \tag{2.2.32}$$

$c(\vec{p}, s)$ and $d(\vec{p}, s)$ are complex quantities that simply denote the coefficients in this expansion. In the quantum theory, $d(\vec{p}, s)^*$ becomes the creation operator for particles and $c(\vec{p}, s)$ the anihilation operator for antiparticles. Their complex conjugate would denote the opposite actions, namely $d(\vec{p}, s)$ would become anihilator of particles and $c(\vec{p}, s)^*$ a creator of antiparticles. An antiparticle and a particle are almost identical, the only difference is that an antiparticle has opposite charges.

By inserting (2.2.32) in the Dirac equation, one can find the explicit expression for the vectors $\boldsymbol{u}(\vec{p}, s)$ and $\boldsymbol{v}(\vec{p}, s)$. We just write the final result here (a proof can be found in [14]):

$$\boldsymbol{u}(\vec{p}, \pm) = \begin{pmatrix} \sqrt{E \mp |\vec{p}|}\boldsymbol{\xi}(\pm) \\ i\sqrt{E \pm |\vec{p}|}\boldsymbol{\xi}(\pm) \end{pmatrix}, \qquad \boldsymbol{v}(\vec{p}, \pm) = \begin{pmatrix} \sqrt{E \pm |\vec{p}|}\boldsymbol{\eta}(\pm) \\ -i\sqrt{E \mp |\vec{p}|}\boldsymbol{\eta}(\pm) \end{pmatrix}. \tag{2.2.33}$$

[7] For any field theory, the number of degrees of freedom is infinite. What we are really counting here is the number of degrees of freedom per spatial point.

[8] After quantization, one can associate two of the degrees of freedom labelled by s with spin up and spin down states. The other two degrees of freedom are associated with particles and antiparticles.

We have used $\pm$ for the index s. $\boldsymbol{\xi}(\pm)$ and $\boldsymbol{\eta}(\pm)$ are simply two-component arbitrary spinors. We can interpret them as the defining spin states of the particles. For example $\boldsymbol{\xi}^T(+) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ would represent a particle with spin up along the 3-axis. It is often convenient to choose spin states to go along the direction of motion of the particle, i.e. to be eigenstates of helicity. Thus,

$$h\boldsymbol{\xi}(\pm) = \pm\frac{1}{2}\boldsymbol{\xi}(\pm). \tag{2.2.34}$$

We also choose

$$\boldsymbol{\eta}(\pm) = -\sigma_2\boldsymbol{\xi}^*(\pm). \tag{2.2.35}$$

Let us show that $\boldsymbol{\eta}(\pm)$ are also eigenstates of the helicity, $h\boldsymbol{\eta}(\pm) = \mp\frac{1}{2}\boldsymbol{\eta}(\pm)$.

***Proof*.** We will make use of the identity $\sigma_2\vec{\sigma}^* = -\vec{\sigma}\sigma_2$. We compute

$$\begin{aligned}(\vec{\sigma}\cdot\vec{p})\ \boldsymbol{\eta}(\pm) &= -p_i\sigma_i\sigma_2\boldsymbol{\xi}(\pm)^* = p_i\sigma_2\sigma_i^*\boldsymbol{\xi}(\pm)^* = \sigma_2(\vec{\sigma^*}\cdot\vec{p})\boldsymbol{\xi}(\pm)^* \\ &= \pm\sigma_2|\vec{p}|\boldsymbol{\xi}(\pm)^* = \mp|\vec{p}|(-\sigma_2\boldsymbol{\xi}(\pm)^*) = \mp|\vec{p}|\boldsymbol{\eta}(\pm). \end{aligned}\tag{2.2.36}$$

And after dividing by $2|\vec{p}|$ we get the desired result. □

For the massless case, where $E = |\vec{p}|$, the spinor $\boldsymbol{u}$ is greatly simplified

$$\boldsymbol{u}(\vec{p},-) = \sqrt{2E}\begin{pmatrix}\boldsymbol{\xi}(-)\\ 0\end{pmatrix}, \qquad \boldsymbol{u}(\vec{p},+) = \sqrt{2E}\begin{pmatrix}0\\ i\boldsymbol{\xi}(+)\end{pmatrix}. \tag{2.2.37}$$

Similarly, for massless spinors $\boldsymbol{v}$,

$$\boldsymbol{v}(\vec{p},-) = \sqrt{2E}\begin{pmatrix}0\\ -i\boldsymbol{\eta}(-)\end{pmatrix}, \qquad \boldsymbol{v}(\vec{p},+) = \sqrt{2E}\begin{pmatrix}\boldsymbol{\eta}(+)\\ 0\end{pmatrix}. \tag{2.2.38}$$

2.2.4 $U(1)$ symmetry for Dirac spinors

Let us consider a global $U(1)$ phase transformation on the Dirac field,

$$\Psi(x) \to \Psi'(x) \equiv e^{i\theta}\Psi(x). \tag{2.2.39}$$

Notice that the transformation for the adjoint field is then $\bar{\Psi} = i\gamma^0\Psi'^\dagger = e^{-i\theta}\bar{\Psi}$ and because of that $\bar{\Psi}'\Psi' = \bar{\Psi}\Psi$. With this, we clearly see that the free Dirac action in (2.2.20) is invariant under this phase transformation.

We compute the Noether current associated to this one-parameter transformation, considering the general formula (B.1.11). Because of the invariance of the action, we see that $K_\mu = 0$. On the other hand $\partial\mathcal{L}/\partial(\partial_\mu\Psi) = -\bar{\Psi}\gamma^\mu$. Thus the conserved Noether current is

$$J^\mu = i\bar{\Psi}\gamma^\mu\Psi. \tag{2.2.40}$$

The time component is given by $J^0 = \Psi^\dagger\Psi$. Precisely, one of Dirac's original motivations for his equation was that, unlike the Klein-Gordon equation, the quantity J^0 could be seen as a positive probability density.

2.3 Majorana spinors

Let us study now Majorana fields, which are Dirac fields that satisfy a reality condition. This restriction reduces the number of degrees of freedom by a factor of 2. Thus, like Weyl fields, a Majorana spinor field is a more fundamental object than a Dirac spinor.

2.3.1 Definition and properties

Firstly we introduce a definition for row vector different from the Dirac adjoint in (2.2.19), called the Majorana conjugate, which sometimes is more convenient. The *Majorana conjugate* of any spinor Ψ is defined as

$$\bar{\Psi} \equiv \Psi^T C. \tag{2.3.1}$$

In order to avoid confusion between Majorana and Dirac conjugate, we may write sometimes $\bar{\Psi}_{\text{Dirac}}$ and $\bar{\Psi}_{\text{Major}}$. The matrix C is called the *charge conjugation matrix* and its definition and properties are discussed in Appendix D.1.3. This matrix has mathematical importance, as it establishes the symmetries of the γ-matrices and also aids raising and lowering spinor indices. But it is also important from the physical point of view, because as we will see, it helps us to relate particles with antiparticles.

For immediate purposes, we only need to make use of the relations

$$C^T = -t_0 C, \qquad \gamma^{\mu T} = t_0 t_1 C \gamma^\mu C^{-1}. \tag{2.3.2}$$

Here t_0 and t_1 can only take the values $\pm$. These values depend on the spacetime dimension D. Let us see what happens if we impose that the Majorana conjugate is equal to the Dirac adjoint

$$\bar{\Psi}_{\text{Major}} = \bar{\Psi}_{\text{Dirac}} \quad \Longrightarrow \quad \Psi^T C = i\Psi^\dagger \gamma^0. \tag{2.3.3}$$

Using (2.3.2) we can rearrange (2.3.3) as

$$\Psi = -it_1 \gamma^0 C^{-1} \Psi^* = -(t_0 t_1) B^{-1} \Psi^*, \tag{2.3.4}$$

where we have introduced the inverse of the matrix $B \equiv it_0 C\gamma_0$. This matrix satisfies

$$\gamma^{\mu *} = -t_0 t_1 B \gamma^\mu B^{-1}, \qquad B^* B = -t_1 \mathbb{1}. \tag{2.3.5}$$

We have proved these identities at the end of Appendix D.1.3. This matrix B is needed to introduce the *charge conjugate* of a spinor, defined as $\Psi^C \equiv B^{-1}\Psi^*$. As we discuss in Appendix D.2.4, the operation of charge conjugation generalizes complex conjugation, and the definition of B is consistent with the complex conjugation properties. Notice that Ψ^C will contain $d(\vec{p}, s)$ and $c(\vec{p}, s)^*$, so we can anticipate that

ψ^C describes antiparticles. We are now ready to talk about Majorana spinors. A *Majorana spinor* is a Dirac spinor satisfying the reality restriction

$$\boxed{\Psi = \Psi^C = B^{-1}\Psi^*, \quad \text{i.e.} \quad \Psi^* = B\Psi}. \tag{2.3.6}$$

This is exactly what we have in (2.3.4) provided that $-(t_0 t_1) = +1$. Furthermore, if we take the complex conjugate in (2.3.6) we see that $\Psi = B^*\Psi^* = B^*B\Psi$, so this reality condition is only consistent if $B^*B = \mathbb{1}$. From (2.3.5), we see that this requires $t_1 = -1$ and thus $t_0 = 1$. Having a glance at the Table of AppendixD.1.3, this happens only for dimensions $D = 2, 3, 4 \bmod 8$. This explains why Majorana spinors can only exist in certain dimensions.

As we see, the Majorana conjugate and the Dirac adjoint are equivalent operations when acting on a Majorana spinor. It is worth noting that the reality condition (2.3.6) does not imply in general that a Majorana spinor has real components. However there are representations in which the γ-matrices are explicitely real. For example, a real representation for $D = 4$ is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{2.3.7}$$

In such representations, $\gamma^{\mu *} = \gamma^\mu$ and because of (2.3.5) we have $B = \mathbb{1}$. Therefore in these representations the Majorana spinor is real as the Majorana condition (2.3.6) becomes $\Psi^* = \Psi$. Moreover, if $B = \mathbb{1}$ then $C = i\gamma^0$. We have already discussed many different operations applied to spinors as well as several spinor types, so we include them in the diagram 2.3.1 for more clarity.

Now we are going to prove that, if $\Psi(x)$ satisfies the free Dirac equation $\partial\!\!\!/\Psi = m\Psi$ for $D = 4$, then the charge conjugate field Ψ^C satisfies the same equation.

Proof. First notice that, as $\partial\!\!\!/\Psi = m\Psi$, the complex conjugate of this equation is $(\gamma^\mu)^*\partial_\mu\Psi^* = m\Psi^*$. Using that $(\gamma^\mu)^* = B\gamma^\mu B^{-1}$ for $D = 4$, we compute

$$\partial\!\!\!/\Psi^C = \gamma^\mu\partial_\mu(B^{-1}\Psi^*) = B^{-1}B\gamma^\mu B^{-1}\partial_\mu\Psi^* = B^{-1}(\gamma^\mu)^*\partial\Psi^* = mB^{-1}\Psi^* = m\Psi^C.$$

□

In Appendix D.2.4 we have also proved that ψ and ψ^C transform in the same way under Lorentz transformations, so the Majorana condition is compatible with Lorentz covariance.

The Majorana condition(2.3.6) implies that Majorana particles are their own anti-particles, mathematically expressed as $c(\vec{p}, s) = d(\vec{p}, s)$. Before showing this, we need to prove that $\boldsymbol{v} = \boldsymbol{u}^C = B^{-1}\boldsymbol{u}^*$ for the functions $\boldsymbol{u}$ and $\boldsymbol{v}$ appearing in the expansion (2.2.32).

We compute B^{-1} for $D = 4$, using the representations (D.1.33) and (D.1.1) that can be found in Appendix D:

$$B^{-1} = i\gamma^0 C^{-1} = i\gamma^0 C^\dagger = i(i\sigma_1 \otimes \mathbb{1})(\sigma_1 \otimes \sigma_2) = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{2.3.8}$$

Then, remembering the choice for $\boldsymbol{\eta}$ in (2.2.35), we have

$$B^{-1}\boldsymbol{u}^* = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \begin{pmatrix} \sqrt{E - |\vec{p}|}\boldsymbol{\xi}^* \\ i\sqrt{E + |\vec{p}|}\boldsymbol{\xi}^* \end{pmatrix} = \begin{pmatrix} \sqrt{E - |\vec{p}|}\boldsymbol{\eta} \\ i\sqrt{E + |\vec{p}|}\boldsymbol{\eta} \end{pmatrix} = \boldsymbol{v}. \tag{2.3.9}$$

Now we apply the reality condition to the Dirac spinor, $(\Psi_+)^C + (\Psi_-)^C = \Psi_+ + \Psi_-$. We compute $(\Psi_+)^C$,

$$(\Psi_+)^C = B^{-1}\Psi_+^* = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E} e^{-i(\vec{p}\cdot\vec{x} - Et)} \sum_s \underbrace{B^{-1}\boldsymbol{u}^*(\vec{p}, s)}_{\boldsymbol{v}(\vec{p},s)} c(\vec{p}, s)^*. \tag{2.3.10}$$

As both $\Psi_\pm$ and $(\Psi_\pm)^C$ are linearly independent, we can identify $(\Psi_+)^C$ with the other term that contains a negative exponential, i.e. Ψ_-:

$$\Psi_- = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E} e^{-i(\vec{p}\cdot\vec{x} - Et)} \sum_s \boldsymbol{v}(\vec{p}, s) d(\vec{p}, s)^*. \tag{2.3.11}$$

From here we can conclude $c(\vec{p}, s)^* = d(\vec{p}, s)^*$ or equivalently $c(\vec{p}, s) = d(\vec{p}, s)$. Since in the quantized theory, $d^*(\vec{p}, s)/c^*(\vec{p}, s)$ become the creation operators of particles/antiparticles, this proves that Majorana particles are their own anti-particles.

2.3.2 Majorana action

Majorana and Dirac fields obey the same equation of motion, namely the Dirac equation. Moreover, Majorana spinors have half of the degrees of freedom of a Dirac fermion, so the action is written as

$$S[\Psi] = -\frac{1}{2} \int \mathrm{d}^D x \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi. \tag{2.3.12}$$

Because of the new barred spinor, $\bar{\psi} = \psi^T C$, we see that the mass and kinetic terms are proportional to $\psi^T C \psi$ and $\psi^T C \gamma^\mu \partial_\mu \Psi$, respectively. Let us suppose that the field components commute. Since C is antisymmetric, the mass term vanishes:

$$\bar{\Psi}\Psi = \Psi^T C \Psi = -\Psi^T C^T \Psi = -\Psi^T \bar{\Psi}^T = -(\bar{\Psi}\Psi)^T = -\bar{\Psi}\Psi \;\rightarrow\; \bar{\Psi}\Psi = 0. \tag{2.3.13}$$

On the other hand, $C\gamma^\mu$ is symmetric, and so the kinetic term is a total derivative:

$$\bar{\Psi}\gamma^\mu \partial_\mu \Psi = \Psi^T C \gamma^\mu \partial_\mu \Psi = \Psi^T (C\gamma^\mu)^T \partial_\mu \Psi = (\partial_\mu \Psi^T C \gamma^\mu \Psi)^T = \gamma^\mu \partial_\mu \bar{\Psi}\Psi = \frac{\gamma^\mu}{2} \partial_\mu (\bar{\Psi}\Psi).$$

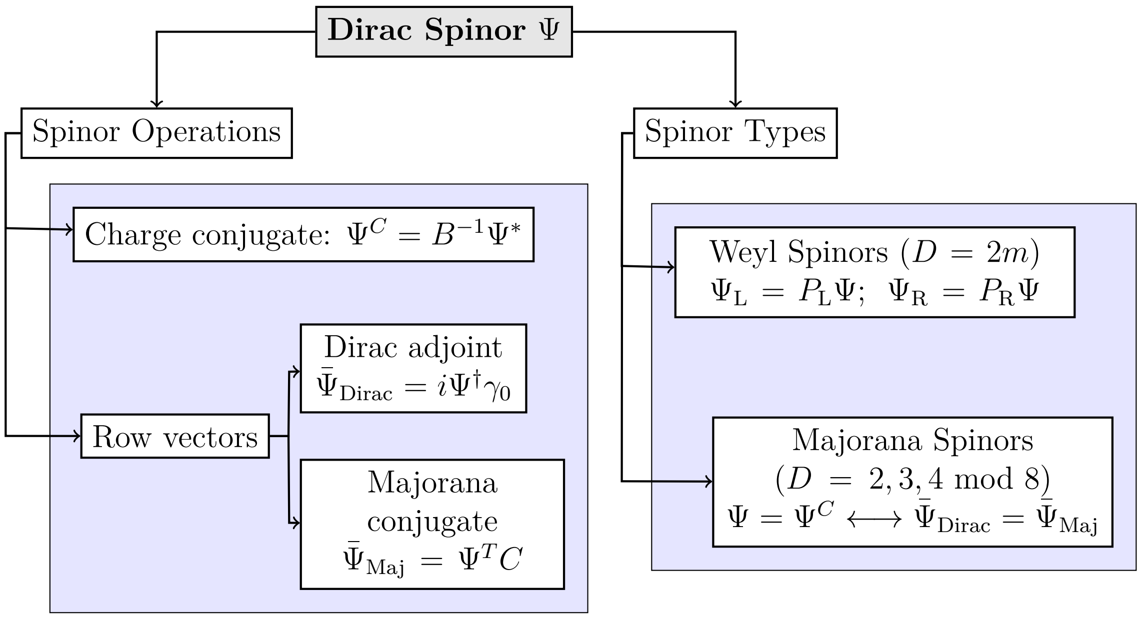

Figure 2.2: *Definitions for some of the different spinor operations and spinor types.*

Thus, the kinetic term is zero when integrated in the action. For commuting field components, there is no dynamics! In order to recover the physical situation, we must assume that Majorana fields are anti-commuting Grassmann variables.

In dimensions $D = 2, 4$ mod 8, both Majorana and Weyl fields can exist. In fact the physics described by them is equivalent since we can write the Lagrangian density of the theory in terms of either fields. Let us show this for $D = 4$. We can rewrite the action (2.3.12) using the chiral projectors P_L and P_R:

$$\begin{aligned} S[\Psi] &= -\frac{1}{2}\int \mathrm{d}^4x\ [\bar{\Psi}\gamma^\mu\partial_\mu - \bar{\Psi}m](P_L + P_R)\Psi = \\ &= -\int \mathrm{d}^4x\ \left[\frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi + \frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu P_R\Psi - \frac{1}{2}m\bar{\Psi}P_L\Psi - \frac{1}{2}m\bar{\Psi}P_R\Psi\right]. \end{aligned} \tag{2.3.14}$$

We are going to manipulate the term $\frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu P_R\Psi$ to see that it is identical to $\frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi$. We compute,

$$\begin{aligned} \frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu P_R\Psi &= \cancel{\frac{1}{2}\partial_\mu\left(\bar{\Psi}\gamma^\mu P_R\Psi\right)} - \frac{1}{2}\partial_\mu\bar{\Psi}\gamma^\mu P_R\psi = -\frac{1}{4}\partial_\mu\bar{\Psi}\gamma^\mu\Psi + \frac{1}{4}\partial_\mu\bar{\Psi}\gamma^\mu\gamma_5\Psi \\ &= \frac{1}{4}\bar{\Psi}\gamma^\mu\partial_\mu\Psi + \frac{1}{4}\bar{\Psi}\gamma^\mu\gamma_5\partial_\mu\Psi = \frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi. \end{aligned} \tag{2.3.15}$$

We have neglected the total derivative term because it vanishes under the integral, and then we have decomposed P_R and used the Majorana flip relation (see (D.2.3) in the Appendix). Thus, the action can be written as

$$S[\Psi] = -\int \mathrm{d}^4x\ \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \frac{1}{2}m\bar{\Psi}P_L\Psi - \frac{1}{2}m\bar{\Psi}P_R\Psi\right]. \tag{2.3.16}$$

Chapter 3

The Maxwell and Yang-Mills gauge fields

In physical theories, invariance under global transformations (those that do not depend on space and time) is important, because it leads to conserved quantities, such as electric charge or isospin. If the invariance is further required under *local* transformations, that do depend on space and time, *interactions can be introduced*. The resulting theories are called gauge theories and they are the core of the Standard Model of particle physics.

Quantum electrodynamics, the quantum version of Maxwell's theory of electromagnetism, is a gauge theory with an Abelian symmetry group $U(1)$. This was the first field theory to be quantized and it has led to some of the most accurate predictions in physics [18]. In 1954, Chen Ning Yang and Robert Mills generalized gauge theories to non-Abelian symmetry groups, in order to explain strong interactions [19].

3.1 The Abelian gauge field

We have already discussed the global $U(1)$ symmetry of free complex scalar and free spinor fields, in Sections 1.2.1 and 2.2.4, respectively. We generalize this situation by considering that the parameter θ becomes an arbitrary function of space and time, $\theta \to \theta(x)$. Therefore we now have an Abelian gauge transformation, consisting of a local change of phase. For example, for a Dirac spinor field, the gauge transformation is implemented as

$$\Psi(x) \ \to \ \Psi'(x) = e^{i\theta(x)}\Psi(x). \tag{3.1.1}$$

In contrast with global phase transformations, the Dirac and the Klein-Gordon actions are not invariant under the transformation (3.1.1), so equations of motion are not gauge invariant. In order to formulate field equations that are gauge invariant, we need to introduce a field $A_\mu(x)$, which is defined to transform as

$$A_\mu(x) \ \to \ A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x). \tag{3.1.2}$$

We have included a numeric factor e, whose meaning will be explained soon. The vector field $A_\mu(x)$ [1], also called *gauge potential*, enters the covariant derivative, defined as follows:

$$D_\mu \Psi(x) \equiv (\partial_\mu - ieA_\mu(x))\Psi(x). \tag{3.1.3}$$

This covariant derivative transforms with the same phase factor as $\Psi(x)$:

$$\begin{aligned} D'_\mu \Psi'(x) =& (\partial_\mu - ieA'_\mu(x))e^{i\theta(x)}\Psi(x) = (\partial_\mu - ieA_\mu(x) - i\partial_\mu\theta(x))e^{i\theta(x)}\Psi(x) \\ =& e^{i\theta(x)}\partial_\mu\Psi(x) + \cancel{\Psi(x)i\partial_\mu\theta(x)e^{iq\theta(x)}} - ieA_\mu(x)e^{iq\theta(x)} - \cancel{\Psi(x)i\partial_\mu\theta(x)e^{iq\theta(x)}} \\ =& e^{iq\theta(x)}D_\mu\Psi(x). \end{aligned} \tag{3.1.4}$$

If we replace $\partial_\mu\Psi(x) \to D_\mu\Psi(x)$ in the free Dirac equation, we get

$$\left[\gamma^\mu D_\mu - m\right]\Psi \equiv \left[\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\Psi = 0. \tag{3.1.5}$$

This equation is gauge covariant: if $\Psi(x)$ satisfies (3.1.5) with $A_\mu(x)$ then $\Psi'(x)$ satisfies the same equation with $A'_\mu(x)$

$$\gamma^\mu D'_\mu \Psi' - m\Psi' = e^{i\theta(x)}\underbrace{\left[\gamma^\mu D_\mu - m\right]\Psi}_{=0} = 0. \tag{3.1.6}$$

The procedure is the same for the complex scalar field $\phi(x)$. The local gauge transformation is $\phi(x) \to \phi'(x) = e^{i\theta(x)}\phi(x)$ becomes a symmetry by defining the covariant derivative $D_\mu\phi(x) \equiv (\partial_\mu - ieA_\mu(x))\phi(x)$ and modifying the Klein-Gordon equation as follows:

$$[D^\mu D_\mu - m^2]\phi(x) = 0. \tag{3.1.7}$$

Therefore we have seen that, by simply promoting the global symmetry to be local, we require the presence of a new vector field $A_\mu(x)$, that couples to $\Psi(x)$ and $\phi(x)$.

The quantization of the field $A_\mu(x)$ leads to a description of massless particles with helicities ± 1, called photons, as it is discussed in [21]. Thus $A_\mu(x)$ represents a bosonic field. Since $A_\mu(x)$ is a vector field, it transforms under the representation $(\frac{1}{2}, \frac{1}{2})$, according to the (j, j') classification of the Lorentz group that we discussed in section 2.1. We have already talked about scalar, spinor and vector fields, being all of them classifiable in the (j, j') representation, so we include them in the Table 3.1. Just for completness, we have also added the (j, j') representation of the metric tensor $g_{\mu\nu}$ and the Rarita-Schwinger field Ψ_μ, which describe the graviton and gravitino, respectively. These are two-hypothetical particles not discovered yet that play an important role in Supergravity theories.

[1] We will assume that A_μ transforms as a vector under Lorentz transformations, but this is an oversimplification, because the question is more subtle. A_μ is undetermined due to the gauge freedom in (3.1.2) and one can eliminate this ambiguity by choosing a certain θ. There are different choices, and A_μ does not transform as vector in all of them. In [20], it is shown that A_μ transforms as a vector in the so-called Lorenz gauge.

Lorentz rep.	**Total spin**	**Mathematical Field**	**Elementary particle**
$(0,0)$	0	Scalar ϕ	Higgs boson
$\left(0,\frac{1}{2}\right)$	$\frac{1}{2}$	Left-chiral spinor Ψ_L	Neutrino
$\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$	$\frac{1}{2}$	Dirac spinor Ψ	Electron, quarks
$\left(\frac{1}{2},\frac{1}{2}\right)$	1	Gauge vectors A_μ, A_μ^A	Photon, gluons
$\left(\frac{1}{2},1\right)\oplus\left(1,\frac{1}{2}\right)$	$\frac{3}{2}$	Rarita-Schwinger Ψ_μ	Gravitino (no SM particle)
$(1,1)$	2	Metric tensor $g_{\mu\nu}$	Graviton (no SM particle)

Table 3.1: *Common representations of the Lorentz group and their corresponding particles. The graviton and gravitino are not included in the Standard Model of particle physics.*

3.1.1 The free case

Although we have introduced the gauge potential $A_\mu(x)$ in order to write the dynamics of the spinor and scalar fields in a gauge invariant fashion, $A_\mu(x)$ can evolve independently, that is, without the presence of any spinor or scalar field. In this section we proceed to derive the dynamical equations of the free gauge field.

We introduce the **field strength**, an antisymmetric derivative of gauge potentials

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{3.1.8}$$

This is a tensor of rank 2. Moreover, the field strength is invariant under gauge transformations, $F'_{\mu\nu} = F_{\mu\nu}$, as the terms $\partial_\mu\partial_\nu\theta$ and $\partial_\nu\partial_\mu\theta$ cancel out. In four dimensions $F_{\mu\nu}$ has six independent components, which we identify with the three components $i = 1, 2, 3$ of the electric field, $E_i = F_{i0}$, and the three components of the magnetic field, $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$.

We look for second order Lorentz covariant equations describing A_μ. We would like to make use of $F_{\mu\nu}$, which is gauge invariant, so we will construct these equations in terms of the first derivatives of $F_{\mu\nu}$. We are going to see that the contracted form

$$\partial^\mu F_{\mu\nu} = 0 \tag{3.1.9}$$

is the suitable choice for the equations of motion of the free electromagnetic field. The strength tensor also satisfies the *Bianchi identity*

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{3.1.10}$$

This equation is satisfied for any $F_{\mu\nu}$ expressed in terms of A_μ as in (3.1.8) (notice that because of $\partial_\mu\partial_\nu A_\rho = \partial_\nu\partial_\mu A_\rho$ all terms in (3.1.10) cancel by pairs). We see that

(3.1.9) and (3.1.10) are tensorial equations, i.e., they hold in any inertial reference system and thus they are Lorentz covariant, as we wished. When these equations are expressed in terms of the electric and magnetic fields, we recover classical Maxwell's equations in the absence of currents and charges [22].

Making use only of (3.1.9) and (3.1.10) we are going to show that the components of the field strength satisfy the wave equation $\Box F_{\mu\nu} = 0$.

Proof. We apply ∂^μ to equation (3.1.10). Taking into account (3.1.9) and the fact that $F_{\mu\nu}$ is antisymmetric, it is also true that $\partial^\mu F_{\nu\mu} = 0$. With this, we have:

$$\partial^\mu \partial_\mu F_{\nu\rho} + \partial^\mu \partial_\nu F_{\rho\mu} + \partial^\mu \partial_\rho F_{\mu\nu} = \partial^\mu \partial_\mu F_{\nu\rho} + \partial_\nu \underbrace{\partial^\mu F_{\rho\mu}}_{=0} + \partial_\rho \underbrace{\partial^\mu F_{\mu\nu}}_{=0} = \Box F_{\nu\rho} = 0.$$

□

The gauge invariant equation $\Box F_{\mu\nu} = 0$ expresses the fact that the electromagnetic field describes massless particles. It is worth noting that the field strength arises as a consequence of the non-commutativity of the covariant derivatives (3.1.3)

$$\begin{aligned} [D_\mu, D_\nu]\Psi =& D_\mu(\partial_\nu \Psi - ieA_\nu \Psi) - D_\nu(\partial_\mu \Psi - ieA_\mu \Psi) \\ =& - ie\partial_\mu(A_\nu \Psi) - ieA_\mu \partial_\nu \Psi + ie\partial_\nu(A_\mu \Psi) + ieA_\nu \partial_\mu \Psi \\ =& - ie\partial_\mu A_\nu \Psi + ie\partial_\nu A_\mu \Psi = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\Psi = -ieF_{\mu\nu}\Psi. \end{aligned} \quad (3.1.11)$$

3.1.2 Dirac field as a source

To account the presence of sources, (3.1.9) has to be modified in the following manner:

$$\boxed{\partial^\mu F_{\mu\nu} = -J_\nu}. \quad (3.1.12)$$

The source J_ν is called the electric current vector. Since $\partial^\nu \partial^\mu F_{\mu\nu}$ vanishes identically [2], the current must be conserved

$$\partial^\nu J_\nu = 0. \quad (3.1.13)$$

The continuity equation (3.1.13) simply expresses the fact that electric charge cannot be created or destroyed. With this, one can show that J_ν actually transforms as a vector. Now, using that $F_{\mu\nu}$ transforms as $F'_{\alpha\beta} = \Lambda_\alpha{}^\mu F_{\mu\nu} \Lambda^\nu{}_\beta$, we can check that equation (3.1.13) is Lorentz covariant

$$\partial'^\alpha F'_{\alpha\beta} = \underbrace{(\Lambda^{-1})_\tau{}^\alpha \Lambda_\alpha{}^\mu}_{\delta^\mu_\tau} \Lambda^\nu{}_\beta \partial^\tau F_{\mu\nu} = \Lambda^\nu{}_\beta \underbrace{\partial^\mu F_{\mu\nu}}_{-J_\nu} = -\Lambda^\nu{}_\beta J_\nu = -J'_\beta. \quad (3.1.14)$$

[2] $\partial^\nu \partial^\mu F_{\mu\nu} = 0$ is a mathematical identity because it is the contraction of a symmetric tensor $\partial^\nu \partial^\mu$ with an antisymmetric one $F_{\mu\nu}$, so an explicit expression for A_μ is not needed.

The current vector J_ν may represent any piece of laboratory equipment, such as a magnetic solenoid. However, from the theoretical physics point of view, it is far more interesting to consider as a source the field Ψ of an elementary charged particle, like the electron. After quantization, this leads to the theory of Quantum Electrodynamics (QED), which describes all electromagnetic phenomena happening in Nature. We proceed to see the classical version for the action functional of QED:

$$S[A_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \ \mathcal{L} = \int \mathrm{d}^D x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \bar{\Psi} \left(\gamma^\mu D_\mu - m \right) \Psi \right]. \tag{3.1.15}$$

This action can be equivalently rewritten as

$$S[A_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{Interaction}} \right], \tag{3.1.16}$$

where $\mathcal{L}_{\text{Dirac}} = -\bar{\Psi}\gamma^\mu \partial_\mu \Psi + m\bar{\Psi}\Psi$ and $\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ describe the dynamics of the spin-$\frac{1}{2}$ particle and the photon, respectively. The other term,

$$\mathcal{L}_{\text{Interaction}} = e\bar{\Psi}\gamma^\mu A_\mu \Psi, \tag{3.1.17}$$

represents the interaction between them. Now we can understand the meaning of e, which is called the *coupling constant*: it measures the strength of the coupling between the photon and the charged particle. The factor $e^2/4\pi \simeq 1/137$ is called the fine structure constant. We now proceed to derive the equations of motion. Firstly, we compute each of the terms appearing in the functional derivative $\delta S / \delta \bar{\Psi}$:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} \right) = \gamma^\mu \partial_\mu \Psi, \qquad \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = m\Psi + ieq\gamma^\mu A_\mu \Psi.$$

With this, we arrive at the gauge covariant Dirac equation of (3.1.5)

$$\frac{\delta S}{\delta \bar{\Psi}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = \left[\gamma^\mu D_\mu - m \right] \Psi = 0. \tag{3.1.18}$$

The functional derivative with respect to the gauge potential A_ν is given by

$$\frac{\delta S}{\delta A^\nu} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0. \tag{3.1.19}$$

We compute each term separately:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = -\left(\partial^\mu A_\nu - \partial_\nu A^\mu \right) \quad \to \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) = -\partial^\mu F_{\mu\nu}, \tag{3.1.20}$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = ieq\bar{\Psi}\gamma_\nu \Psi. \tag{3.1.21}$$

The resulting equation of motion is thus

$$\partial^\mu F_{\mu\nu} = -ieq\bar{\Psi}\gamma_\nu \Psi. \tag{3.1.22}$$

This is the same as (3.1.12) with the electric current proportional to the Noether current of the global $U(1)$ phase symmetry discussed in Section 2.2.4. Equations (3.1.18) and (3.1.22) determine both fields Ψ and A_μ. The former equation tells that the dynamics of Ψ is affected by the field A_μ, whereas the later tells that Ψ acts as the same time as a source for A_μ.

3.1.3 Energy-momentum tensor

We consider in (3.1.16) only the terms describing the free electromagnetic field, that is, $\mathcal{L}_{\text{Maxwell}}$. This action is invariant under spacetime translations. Proceeding as we did in Section (1.2.2), we find that the energy-momentum tensor is given by

$$J^\mu{}_\nu = T^\mu{}_\nu = K^\mu{}_\nu - \frac{\partial \mathcal{L}_{\text{Maxwell}}}{\partial(\partial_\mu A_\rho)}\partial_\nu A_\rho = \delta^\mu_\nu \mathcal{L}_{\text{Maxwell}} + F^{\mu\rho}\partial_\nu A_\rho. \tag{3.1.23}$$

Raising the ν index with the help of the metric and writing $\mathcal{L}_{\text{Maxwell}}$ explicitely, we have

$$T^{\mu\nu} = -\frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\rho}\partial^\nu A_\rho. \tag{3.1.24}$$

Because of the presence of $\partial^\nu A_\rho$, this expression for the energy-momentum tensor is not gauge invariant. Gauge symmetry can be restorted by adding the derivative of an antisymmetric tensor to this Noether current (see Appendix B.1.2). We add the term $\partial_\rho(A^\nu F^{\rho\mu})$ to (3.1.24), so:

$$\begin{aligned} T'^{\mu\nu} = T^{\mu\nu} + \partial_\rho(A^\nu F^{\rho\mu}) &= -\frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\rho}\partial^\nu A_\rho - \partial_\rho A^\nu F^{\mu\rho} \\ &= -\frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\rho}F^\nu{}_\rho. \end{aligned} \tag{3.1.25}$$

Therefore, $T'^{\mu\nu}$ is now gauge invariant. We can check that in four dimensions the elements of (3.1.25) lead to well-known results of classical electromagnetism. For example, using that the elements of the field strength can be expressed in terms of the electric and magnetic field components as $F_{i0} = E_i$ and $F_{ij} = \varepsilon_{ijk}B^k$ respectively, we compute the following:

$$F^{0\rho}F^0{}_\rho = F^{0i}F^0{}_i = E^iE_i = \vec{E}^2, \tag{3.1.26}$$

$$F^{\alpha\beta}F_{\alpha\beta} = 2F^{i0}F_{i0} + F^{ji}F_{ji} = -2E^iE_i + \underbrace{\varepsilon_{ijk}\varepsilon^{jil}}_{2\delta^l_k}B^kB_l = 2(\vec{B}^2 - \vec{E}^2). \tag{3.1.27}$$

In this way we can obtain T'^{00}. As we know, this should represent the energy density. We get:

$$T'^{00} = F^{0\rho}F^0{}_\rho + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} = \vec{E}^2 + \frac{1}{2}(\vec{B}^2 - \vec{E}^2) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2). \tag{3.1.28}$$

This is the classical result for the energy density of the electromagnetic field [22].

3.2 The non-Abelian gauge fields

Yang-Mills theory constitutes a generalization of electromagnetism, in the sense that the symmetry group of the theory is now non-Abelian (as opposed to $U(1)$, which is Abelian). Examples of non-Abelian groups that play an important role in the Standard Model are $SU(2)$ and $SU(3)$. These are discussed in Appendix C.2.1.

In Yang-Mills theory, scalar and spinor fields transform in an irreducible representation R of a non-Abelian Lie group G. As we explain in Appendix C.2, a general element of the group is denoted by $e^{-\theta^A t_A}$, where θ^A are the parameters of the transformation and t_A the generators of the algebra of the group, $\mathfrak{g}$. For example, a set of Dirac spinor fields Ψ^α ($\alpha = 1, ..., \dim R$) [3] transforms as

$$\Psi^\alpha(x) \;\rightarrow\; \left(e^{-\theta^A t_A}\right)^\alpha{}_\beta \Psi^\beta(x). \tag{3.2.1}$$

The set of Dirac conjugate spinors 2.2.19, denoted by $\bar{\Psi}_\alpha$, transforms as

$$\bar{\Psi}_\alpha \;\rightarrow\; \bar{\Psi}_\beta \left(e^{\theta^A t_A}\right)^\beta{}_\alpha. \tag{3.2.2}$$

In general we will only need to consider infinitesimal transformations, given by truncation of the exponential series at first order in θ^A. Omitting α indices, we write:

$$\delta\Psi = -\,\theta^A t_A \Psi, \tag{3.2.3}$$

$$\delta\bar{\Psi} = \bar{\Psi}\theta^A t_A. \tag{3.2.4}$$

We can check that the global transformations in (3.2.1) and (3.2.2) leave the action (2.2.20) for the free Dirac field invariant:

$$\begin{aligned} S'[\bar{\Psi}, \Psi] &= -\int \mathrm{d}^D x \; \bar{\Psi}'[\gamma^\mu \partial_\mu - m]\Psi' \\ &= -\int \mathrm{d}^D x \; \bar{\Psi}\cancel{e^{\theta^A t_A}}[\gamma^\mu \partial_\mu - m]\cancel{e^{-\theta^A t_A}}\Psi = S[\bar{\Psi}, \Psi]. \end{aligned} \tag{3.2.5}$$

Therefore, these transformations constitute a symmetry of the system and we can find the corresponding Noether current. If we identify the general parameters ε^A with θ^A and consider that $\delta\Psi = \varepsilon^A \Delta_A \Psi = -\theta^A t_A \Psi$, then the infinitesimal transformation is

$$\Delta_A \Psi = -t_A \Psi. \tag{3.2.6}$$

Having in mind the general expression in Appendix (B.1.11), we notice that $K^\mu_A = 0$ given the invariance of the Lagrangian density. Thus the Noether current is

$$J_{\mu A} = -\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Psi)} \Delta_A \Psi = -\bar{\Psi} t_A \gamma_\mu \Psi. \tag{3.2.7}$$

[3] Indices α should not be confused with spinor indices, which we normally omit.

The next step in the discussion is to gauge the global symmetry that we have just discussed. That is, we promote the group parameters θ^A to be arbitrary functions of space and time, $\theta^A \to \theta^A(x)$. Field equations for the free spinor Ψ are not invariant anymore under this symmetry, so we need to introduce a set of vector fields $A_\mu^A(x)$, whose infinitesimal transformation is

$$\delta A_\mu^A(x) = \frac{1}{g}\partial_\mu \theta^A(x) + \theta^C(x) A_\mu^B(x) f_{BCA}. \tag{3.2.8}$$

The vectors $A_\mu^A(x)$ are also called *non-Abelian gauge fields*. The constant g is the Yang-Mills coupling, and measures the strength of the interaction, in the same way as the electromagnetic coupling e. The array of numbers $f_{BCA} = -f_{CBA}$ denotes the structure constants of the group (we discuss them in Appendix C.2).

The fields $A_\mu^A(x)$ enter the covariant derivatives, which are defined as

$$D_\mu \Psi = (\partial_\mu + g t_A A_\mu^A)\Psi, \tag{3.2.9}$$
$$D_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} - g\bar{\Psi} t_A A_\mu^A. \tag{3.2.10}$$

Following the ideas of Section 3.1, we can obtain gauge invariant equations of motion if we replace $\partial_\mu \to D_\mu$. The action for the spinor field Ψ would become

$$S = -\int \mathrm{d}^D x\, \mathcal{L} = -\int \mathrm{d}^D x\, [\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi], \tag{3.2.11}$$

which leads to the equation of motion

$$[\gamma^\mu D_\mu - m]\Psi^\alpha = 0. \tag{3.2.12}$$

In order to check this action is gauge invariant, we are going to prove that the infinitesimal gauge transformation of the covariant derivative $D_\mu \Psi$ is the same as the one for the field Ψ. Namely, we are going to show that $\delta D_\mu \Psi = -\theta^A t_A D_\mu \Psi$.

Proof. We will make use of the commutation relation of the Lie algebra, $[t_A, t_D] = f_{ADE} t_E$. We compute:

$$\begin{aligned}
\delta D_\mu \Psi =& \partial_\mu(\delta\Psi) + g t_A \delta A_\mu^A \Psi + g t_A A_\mu^A \delta\Psi \\
=& -\theta^A t_A \partial_\mu \Psi + g\theta^C A_\mu^B t_A f_{BCA}\Psi - g\theta^D A_\mu^A t_A t_D \Psi \\
=& -\theta^A t_A \partial_\mu \Psi - \underbrace{g\theta^D A_\mu^A t_D t_A \Psi}_{D\to A,\ A\to D} + \cancel{g\theta^C A_\mu^B t_A f_{BCA}\Psi} - \underbrace{\cancel{g\theta^D A_\mu^A f_{ADE} t_E \Psi}}_{D\to C,\ A\to B,\ E\to A} \\
=& -\theta^A t_A(\partial_\mu \Psi + g t_D A_\mu^D \Psi) = -\theta^A t_A D_\mu \Psi.
\end{aligned}$$

□

Now we are ready to show that the action (3.2.11) is gauge invariant

$$\begin{aligned}
\delta S =& -\int \mathrm{d}^D x\, [\delta\bar{\Psi}\gamma^\mu D_\mu \Psi + \bar{\Psi}\gamma^\mu \delta D_\mu \Psi - m(\delta\bar{\Psi}\Psi + \bar{\Psi}\delta\Psi)] \\
=& -\int \mathrm{d}^D x\, [\bar{\Psi}\theta^A t_A \gamma^\mu D_\mu \Psi - \bar{\Psi}\gamma^\mu \theta^A t_A D_\mu \Psi - m(\bar{\Psi}\theta^A t_A \Psi - \bar{\Psi}\theta^A t_A \Psi)] = 0.
\end{aligned}$$

3.2.1 Yang-Mills field strength and action

The first part of this section is aimed to find quantities that dictate how the fields $A_\mu^A(x)$ evolve independently. In Section 3.1.1 we saw that the field strength $F_{\mu\nu}$ emerges as a consequence of the non-commutativity of the covariant derivatives. We proceed to compute the commutator of the new covariant derivatives defined in the previous section, to see if we can obtain an object analog to $F_{\mu\nu}$ in a similar fashion:

$$\begin{aligned}
[D_\mu, D_\nu]\Psi =& (\partial_\mu + gt_B A_\mu^B)(\partial_\nu \Psi + gt_A A_\nu^A \Psi) - (\partial_\nu + gt_A A_\nu^A)(\partial_\mu \Psi + gt_B A_\mu^B \Psi) \\
=& gt_A \partial_\mu A_\nu^A \Psi + \cancel{gt_A A_\nu^A \partial_\mu \Psi} + \cancel{gt_B A_\mu^B \partial_\nu \Psi} + g^2 t_B t_A A_\mu^B A_\nu^A \Psi - \\
& \underbrace{-gt_B \partial_\nu A_\mu^B \Psi}_{B\to A} - \cancel{gt_B A_\mu^B \partial_\nu \Psi} - \cancel{gt_A A_\nu^A \partial_\mu \Psi} - g^2 t_A t_B A_\mu^B A_\nu^A \Psi \\
=& g(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A) t_A \Psi + g^2 \underbrace{[t_B, t_A]}_{f_{BAC} t_C} A_\mu^B A_\nu^A \Psi = g F_{\mu\nu}^A t_A \Psi.
\end{aligned} \quad (3.2.13)$$

We have arrived at an expression for the so-called *Yang-Mills field strength*

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f_{BCA} A_\mu^B A_\nu^C. \quad (3.2.14)$$

This antisymmetric tensor is the non-Abelian generalization of the electromagnetic field strength (3.1.8). An important difference with $F_{\mu\nu}$ is that the Yang-Mills strength is not gauge invariant. In fact, one can show that $F_{\mu\nu}$ transforms as a field in the adjoint representation:

$$\delta F_{\mu\nu}^A = \theta^C F_{\mu\nu}^B f_{BCA}. \quad (3.2.15)$$

We have discussed the adjoint representation in Appendix C.2. Another important difference with the electromagnetic case is that $F_{\mu\nu}^A$ is *nonlinear* in A_μ^A.

Let us now formulate the equations governing the dynamics of A_μ^A. We consider the presence of matter sources, described by current vectors J_ν^A. The following equations

$$\boxed{D^\mu F_{\mu\nu}^A = \partial^\mu F_{\mu\nu}^A + g f_{BCA} A^{\mu B} F_{\mu\nu}^C = -J_\nu^A}, \quad (3.2.16)$$

which are both gauge and Lorentz covariant, are the equations of motion we are looking for. These are the Yang-Mills equations, and are analogous to Maxwell's equations (3.1.12). A meaningful difference is that, even in the absence of sources $J_\nu^A = 0$, (3.2.16) is still a complicated non-linear equation with non-trivial solutions for A_μ^A. One can show that $D^\nu D^\mu F_{\mu\nu}^A$ vanishes identically, so the current needs to be conserved under the covariant derivative

$$D^\nu J_\nu^A = 0. \quad (3.2.17)$$

The Yang-Mills field strength satisfies the Bianchi identity:

$$D_\mu F_{\nu\rho}^A + D_\nu F_{\rho\mu}^A + D_\rho F_{\mu\nu}^A = 0, \quad (3.2.18)$$

which is the analog of (3.1.10). We now proceed to prove it.

Proof. We straightforwardly compute each of the three terms in (3.2.18) separately. The first one is

$$\begin{aligned} D_\mu F^A_{\nu\rho} = & \underbrace{\partial_\mu \partial_\nu A^A_\rho}_{1} - \underbrace{\partial_\mu \partial_\rho A^A_\nu}_{2} + \underbrace{g f_{BCA} \partial_\mu A^B_\nu A^C_\rho}_{8} + \underbrace{g f_{BCA} A^B_\nu \partial_\mu A^C_\rho}_{5} + \underbrace{g f_{BCA} A^B_\mu \partial_\nu A^C_\rho}_{10} \\ & - \underbrace{g f_{BCA} A^B_\mu \partial_\rho A^C_\nu}_{4} + \underbrace{g^2 f_{BCA} f_{DEC} A^B_\mu A^D_\nu A^E_\rho}_{7} . \end{aligned} \tag{3.2.19}$$

We enumerate each term in order to distinguish clearly the cancellations. We obtain $D_\nu F^A_{\rho\mu}$ and $D_\rho F^A_{\mu\nu}$ by simply permutating greek indices in a cyclic way

$$\begin{aligned} D_\nu F^A_{\rho\mu} = & \underbrace{\partial_\nu \partial_\rho A^A_\mu}_{3} - \underbrace{\partial_\nu \partial_\mu A^A_\rho}_{1} + \underbrace{g f_{BCA} \partial_\nu A^B_\rho A^C_\mu}_{10} + \underbrace{g f_{BCA} A^B_\rho \partial_\nu A^C_\mu}_{6} + \underbrace{g f_{BCA} A^B_\nu \partial_\rho A^C_\mu}_{9} \\ & - \underbrace{g f_{BCA} A^B_\nu \partial_\mu A^C_\rho}_{5} + \underbrace{g^2 f_{BCA} f_{DEC} A^B_\nu A^D_\rho A^E_\mu}_{7} , \end{aligned} \tag{3.2.20}$$

$$\begin{aligned} D_\rho F^A_{\mu\nu} = & \underbrace{\partial_\rho \partial_\mu A^A_\nu}_{2} - \underbrace{\partial_\rho \partial_\nu A^A_\mu}_{3} + \underbrace{g f_{BCA} \partial_\rho A^B_\mu A^C_\nu}_{9} + \underbrace{g f_{BCA} A^B_\mu \partial_\rho A^C_\nu}_{4} + \underbrace{g f_{BCA} A^B_\rho \partial_\mu A^C_\nu}_{8} \\ & - \underbrace{g f_{BCA} A^B_\rho \partial_\nu A^C_\mu}_{6} + \underbrace{g^2 f_{BCA} f_{DEC} A^B_\rho A^D_\mu A^E_\nu}_{7} . \end{aligned} \tag{3.2.21}$$

All the terms labeled with the same numbers cancel when summed (this becomes clear when a relabeling of latin dummy indices is made). Note that the three terms labeled by 7 vanish when summed (after relabeling of latin indices) because of Jacobi identity (see Appendix C.2). Thus (3.2.18) is satisfied. □

We dedicate the last part of this section to discuss the action functional describing the non-Abelian gauge fields A^A_μ coupled to the set of Dirac fields Ψ^α. The gauge invariant action is

$$S[A^A_\mu, \bar{\Psi}_\alpha, \Psi^\alpha] = \int \mathrm{d}^D x \ \mathcal{L} = \int \mathrm{d}^D x \ \left[-\frac{1}{4} F^{A\mu\nu} F^A_{\mu\nu} - \bar{\Psi}_\alpha (\gamma^\mu D_\mu - m) \Psi^\alpha \right] . \tag{3.2.22}$$

We proceed to derive the equations of motion arising from this action. The functional derivative respect to A^A_μ is given by

$$\frac{\delta S}{\delta A^A_\nu} = \frac{\partial \mathcal{L}}{\partial A^A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^A_\nu)} \right) = 0. \tag{3.2.23}$$

If we define $\bar{F}^A_{\mu\nu} \equiv \partial_\mu A^A_\nu - \partial_\nu A^A_\mu$ and compute each term, we obtain:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^A_\nu)} \right) = -\partial^\mu \bar{F}^A_{\mu\nu} - g f_{BCA} \partial^\mu (A^B_\mu A^C_\nu). \tag{3.2.24}$$

$$\frac{\partial \mathcal{L}}{\partial A^A_\nu} = -g f_{BCA} A^{\mu B} \bar{F}^C_{\mu\nu} - g^2 f_{BCA} f_{DEC} A^{\mu B} A^D_\mu A^E_\nu - g \bar{\Psi}_\alpha \gamma_\nu t_A \Psi^\alpha. \tag{3.2.25}$$

Now, if we substitute the expression (3.2.14) for $F^A_{\mu\nu}$ into (3.2.16), we realize that the different terms we get are precisely those we have just found in (3.2.24) and (3.2.25). Thus, the equation of motion is

$$D^\mu F^A_{\mu\nu} = -gJ_\nu = -g\bar{\Psi}_\alpha \gamma_\nu t_A \Psi^\alpha. \tag{3.2.26}$$

Again, the spinor fields Ψ^α act as a source for the non-Abelian gauge fields. Finally, the functional derivative respect to $\bar{\Psi}_\alpha$ yields

$$\frac{\delta S}{\delta \bar{\Psi}_\alpha} = \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_\alpha} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}_\alpha)} \right) = m\Psi^\alpha - g\gamma^\mu t_A A^A_\mu \Psi^\alpha - \gamma^\mu \partial_\mu \Psi^\alpha = 0. \tag{3.2.27}$$

This is precisely the gauge covariant Dirac equation in (3.2.12).

Chapter 4

Introduction to SUSY

SUSY is a symmetry connecting bosons and fermions. In particular, SUSY proposes that every particle has a partner, called *superpartner*. These superpartners are very similar to standard particles. The main difference is that they have a spin that differs by 1/2 from that of the conventional particles.

SUSY theories first appeared at the beginning of the 1970's. In 1974, Julius Wess and Bruno Zumino studied the first interacting quantum field theory which was invariant under linear supersymmetric transformations [5].

SUSY is an active research field nowdays. The Minimal Supersymmetric Standard Model (MSSM), which constitutes a supersymmetric extension of the Standard Model, solves many current problems in particle physics, such as the naturalness problem or the gauge coupling unification.

4.1 Why SUSY?

SUSY tells us that for each boson/fermion, one should find a fermion/boson with the same mass and the same quantum numbers. The general rule is to prefix with "s" the name of the superpartners of fermions and to suffix with "ino" the name of the superpartners of bosons [1]. For example, the superpartners of the electron and the quarks are called selectron and squarks, respectively. In the same way, the superpartners of the gluon, photon and Higgs particle are called gluino, photino and Higgsino, respectively. In Figure 4.1 we show the SM particles and their superpartners.

However, these superpartners have not been observed in Nature so far. For example, we have not observed any spin-0 particle with the same mass and the same quantum numbers as the electron. Thus, if SUSY exists, it must be spontaneously broken in order to allow superpartners to have bigger masses. This would explain why they have not been detected yet in particle accelerators, such as the Large Hadron

[1] An exception to this rule is the case of the neutrino, whose superpartner is called *neutralino*.

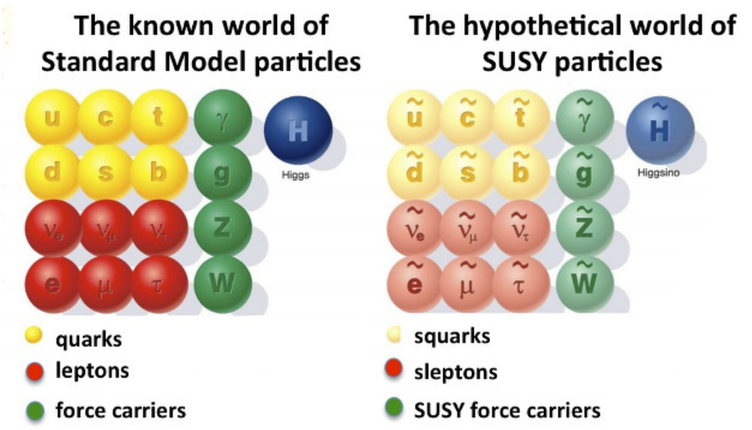

Figure 4.1: The particles of the Standard Model and their corresponding superpartners. Picture taken from [23].

Collider (LHC). But this is not new, since spontaneous symmetry breaking already happens in the SM. For example, the exact symmetry of this theory, described by the direct group product $SU(3) \times SU(2) \times U(1)$, imply that all bosons are massless, in contradiction with the observed masses of the $W^{\pm}$ and Z bosons. One solves this problem by spontaneously breaking the symmetry via the Higgs mechanism.

The MSSM describes spontaneously broken supersymmetry and answers many open questions. Here we mention some motivating reasons for studying SUSY from a phenomenological point of view.

- **Naturalness problem**. The mass of the Higgs boson, whose experimental value is approximately 125 GeV, is very sensitive to quantum corrections, which are estimated to be of the order of 10^{30} GeV. In order to keep the experimental value of 125 GeV, an unnatural fine tuning procedure is required. However, if SUSY is considered, contributions from bosonic and fermionic superpartners cancel exactly the quantum corrections. This was one of the original motivations for developing SUSY.

- **Gauge coupling unification**. There have been some attempts in constructing a Grand Unified Theory (GUT), in which the three interactions of the Standard Model are merged into a single one at high energies [2]. This theory implies a unified coupling constant. The problem is that the couplings in the Standard Model do not seem to intersect (see Figure 4.2). With SUSY, the dependence on the energy scale is modified and the couplings almost unify at the order of 10^{16} GeV.

[2]The variation of the coupling constants with respect to energy scale is determined by the β functions, that are studied in advanced courses of QFT.

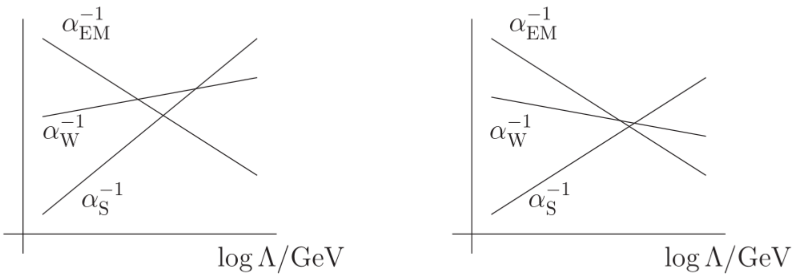

Figure 4.2: The strong, weak and electromagnetic couplings, with and without SUSY. Picture taken from [24].

- **Dark matter**. The total energy contained in the universe is divided into approximately 4 % ordinary matter, 22 % dark matter and 74 % dark energy. Despite the abundance of dark matter in contrast to ordinary matter, we still do not know what are the particles that make up dark matter. However, SUSY particles are natural candidates for dark matter. In particular, the neutralino (the lightest particle of the MSSM) exhibits many of the necessary properties required by experimental evidences.

- **Quantum gravity**. A special feature of SUSY is that internal and spacetime transformations are mixed. Because of this, when invariance is imposed under *local* SUSY transformations (that is, for parameters $\varepsilon_\alpha(x)$ that depend on space and time), one is forced to introduce fields that reproduce General Relativity. The resulting theory is called *supergravity* (SUGRA). Thus, in the same way that gauging the $U(1)$ symmetry leads to electromagnetism, gauging SUSY leads to gravity. The hypothetical elementary particle that mediates gravity is the graviton, a spin-2 particle, and his superpartner is a spin-$\frac{3}{2}$ particle called the gravitino. Besides SUGRA, the supersymmetry algebra is contained in other theories of quantum gravity, such as superstring theory.

- **Further applications**. Apart from theoretical physics, SUSY concepts have been applied in many different areas. For instance, in the field of Integrated Optics, certain branches of SUSY can be explored in accessible laboratory settings [25]. Another example is Condensed matter physics, where SUSY has been applied to disordered quantum systems [26].

4.2 Basic concepts in SUSY field theory

A supersymmetric transformation turns bosons into fermions and fermions into bosons. We can schematically denote particle states of bosons and fermions by $|\text{Boson}\rangle$ and

$|\text{Fermion}\rangle$. Thus, an operator Q that generates supersymmetry transformation must act in the following manner:

$$Q\,|\text{Boson}\rangle = |\text{Fermion}\rangle\,, \qquad Q\,|\text{Fermion}\rangle = |\text{Boson}\rangle\,. \tag{4.2.1}$$

From this, we note that the Q's change the spin, and hence the statistics of the fields. Spin is related to rotations, so we can infer that supersymmetry is, in some sense, related to spacetime transformations. Let us consider a simple example. Let us assume U to be a unitary operator in the Hilbert space which corresponds to a 360° rotation around a particular axis. Then:

$$UQ\,|\text{Boson}\rangle = UQU^{-1}U\,|\text{Boson}\rangle = U\,|\text{Fermion}\rangle\,, \tag{4.2.2}$$

$$UQ\,|\text{Fermion}\rangle = UQU^{-1}U\,|\text{Fermion}\rangle = U\,|\text{Boson}\rangle\,. \tag{4.2.3}$$

However, we do know that fermions and bosons behave differently under rotations:

$$U\,|\text{Fermion}\rangle = -\,|\text{Fermion}\rangle\,, \qquad U\,|\text{Boson}\rangle = |\text{Boson}\rangle\,. \tag{4.2.4}$$

Then we must have:

$$UQU^{-1} = -Q \tag{4.2.5}$$

That is to say, the rotated symmetry generator Q picks a minus sign, just as fermionic states do. This is why Q is a spinor operator. For simplicity, it is assumed that Q is a four-component Majorana spinor, although there are other equivalent treatments which consider two-component Weyl spinors (as in [27]). Therefore, we write in general Q^i_α, where α is a spinor index and $i = 1, ..., \mathcal{N}$ is an index labeling different operators Q^i. Then, the number of Q's will always be a multiple of the spinorial components, and $\mathcal{N}$ determines the multiple. Theories with $\mathcal{N} \geq 2$ are called extended supersymmetric theories. Since the Q^i generate supersymmetry transformations, they are also referred as supercharges (we have discussed charges as generators of transformations in Appendix B.2). Here we consider the simplest case of $\mathcal{N} = 1$. We will also assume dimension $D = 4$.

Generators of transformations have to fulfill a certain algebra, so we can ask ourselves: what is the algebra satisfied by Q? To answer this question, we have to extend the symmetry of generators to a more general structure. So far, all the generators we have seen obey commutation relations. For example, the generators $J_{[\mu\nu]}$ and P_μ of the Poincaré group, which describes spacetime symmetries, satisfy the relations [3]

$$\begin{aligned}
[J_{[\mu\nu]}, J_{[\rho\sigma]}] &= \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{[\nu\sigma]} - \eta_{\nu\sigma} J_{[\mu\rho]} + \eta_{\mu\sigma} J_{[\nu\rho]}, \\
\left[J_{[\rho\sigma]}, P_\mu\right] &= P_\rho \eta_{\sigma\mu} - P_\sigma \eta_{\rho\mu}, \\
[P_\mu, P_\nu] &= 0.
\end{aligned} \tag{4.2.6}$$

[3]These relations are discussed in Appendix C.2.3.

Moreover, the generators T_a of a certain Lie group which contain internal symmetries satisfy in general

$$[T_a, T_b] = f_{abc} T_c, \tag{4.2.7}$$

where f_{abc} are the structure constants of the group. Generators with a commutator structure are said to be bosonic (B). But we can also consider the possibility that they satisfy anticommutation relations, in which case the generators are said to be fermionic (F). The later has to be the case of the generators Q, since they are spinors.

However, there are two important theorems that limitate the type of generators and algebras that can be realized in an interacting relativistic quantum field theory. Concerning bosonic generators we have the **Coleman-Mandula (CM) theorem** [28]. Under the assumption of a discrete spectrum of massive one-particle states with positive energies, this theorem states that the symmetry group G of the theory can only be of the form $G =$ Poincaré group $\times$ Internal Symmetries. That is to say, the associated Lie algebra of G can only be the direct sum of the Poincaré algebra (4.2.6) and the Lie algebra of internal symmetries. Because G is a direct sum, these algebras commute:

$$[T_a, P_\mu] = [T_a, J_{[\mu\nu]}] = 0. \tag{4.2.8}$$

It is worth noting that the assumptions of the CM theorem are satisfied in the Standard Model, and the internal symmetries of the gauge group of the theory, which is $SU(3) \times SU(2) \times U(1)$, do not mix with spacetime symmetries.

Thus, if we were to suggest the possibility of combining spacetime and internal symmetries in a non-trivial way, that is, having for example $[T_a, J_{[\mu\nu]}] \neq 0$, the CM theorem would seem to prevent us of doing so. Nevertheless, there is a hidden assumption in the CM theorem: we assume Lie algebras, which restricts our generators to satisfy commutation relations. If we allow the presence of anticommutators (and thus of fermionic generators) the CM theorem can be avoided. Then the **Haag–Łopuszański–Sohnius (HLS) theorem** [29] comes into play.

The HLS theorem, under the same hypothesis of the CM theorem, states that bosonic and fermionic generators can join in a new structure called *superalgebra* [4]. The schematic structure of the superalgebra is

$$[B, B] = B, \qquad [B, F] = F, \qquad \{F, F\} = B. \tag{4.2.9}$$

[4] A superalgebra or graded Lie algebra is a particular case of a more general mathematical structure, called graded algebra.

In particular, the so-called Super Poincaré algebra, which is a minimal supersymmetric extension of the Poincaré algebra (4.2.6), exhibits this structure:

$$\begin{aligned} &\left\{Q_\alpha, Q^\beta\right\} = 0, && \left\{Q_\alpha, \bar{Q}^\beta\right\} = (\gamma_\mu)_\alpha{}^\beta P^\mu, \\ &[J_{[\mu\nu]}, Q_\alpha] = (\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta, && [P_\mu, Q_\alpha] = 0. \end{aligned} \tag{4.2.10}$$

Here the bar over Q denotes either a Dirac adjoint or a Majorana conjugate (which, as shown in (2.3.3), are equivalent for a Majorana spinor) and $\gamma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}$. So we have finally shown that we have to extend the notion of a Lie algebra to a graded Lie algebra, or superalgebra, to accomodate the fermionic generators.

The parameters of SUSY transformations are constant 4-component Majorana spinors ε_α[5]. As the parameters do not depend on space and time, we will be talking about global SUSY transformations [6]. Note that, since ε_α is not a field, there is no particle associated with it.

Using the canonical formalism (see Appendix B.2), one is able compute the field variations of an arbitrary field $\hat{\Phi}$ once the explicit form of the supercharges is known,

$$\delta\hat{\Phi} = -i[\bar{\varepsilon}^\alpha Q_\alpha, \hat{\Phi}(x)]. \tag{4.2.11}$$

Let us explore some consequences of (4.2.10). We are going to compute the commutator of succesive variations δ_1, δ_2 of $\hat{\Phi}$, with parameters ε_1 and ε_2, respectively. We will use that $\bar{\varepsilon}Q = \bar{Q}\varepsilon$ for Majorana spinors (see (D.2.4) in Appendix D.2.1). We get

$$\begin{aligned} [\delta_1, \delta_2]\hat{\Phi}(x) &= \delta_1(\delta_2\hat{\Phi}) - \delta_2(\delta_1\hat{\Phi}) \\ &= -\left[\bar{\varepsilon}_1 Q, \left[\bar{Q}\varepsilon_2, \hat{\Phi}(x)\right]\right] + \left[\bar{\varepsilon}_2 Q, \left[\bar{Q}\varepsilon_1, \hat{\Phi}(x)\right]\right] \\ &= -\bar{\varepsilon}_1^\alpha \left[\left\{Q_\alpha, \bar{Q}^\beta\right\}, \hat{\Phi}(x)\right] \varepsilon_{2\beta} = -\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\partial_\mu\hat{\Phi}(x). \end{aligned} \tag{4.2.12}$$

In the second line we have used the so-called superJacobi identity [7] and to reach the last line we have made use of the second relation in (4.2.10). Notice that in the right hand side of (4.2.12) we have the expression of an infinitesimal spacetime translation, with parameter $a^\mu = -\bar{\varepsilon}_1\gamma^\mu\varepsilon_2$. We have thus arrived at a remarkable result: after performing two succesive SUSY transformations on a field, we obtain the same field

[5] This is because they normally appear contracted with Q so, in order to have a Lorentz scalar quantity, the only possibility for ε_α is to be also Majorana spinors. In Appendix D.2.1 we show that two spinors $\boldsymbol{\lambda}$ and $\boldsymbol{\chi}$ can form a Lorentz scalar quantity by the contraction $\bar{\boldsymbol{\lambda}}\boldsymbol{\chi}$.

[6] We are not considering the case in which $\partial_\mu\varepsilon_\alpha \neq 0$, which has important implications for the unification of gravity and internal symmetries, as we discussed.

[7] In a superalgebra, for any pair of fermionic operators F_1 and F_2 and a bosonic operator B, the following generalised Jacobi identity holds:

$$[F_1, [F_2, B]] - [F_2, [F_1, B]] = [\{F_1, F_2\}, B].$$

but evaluated at a different coordinate than it was initially. This elucidates that supersymmetry is deeply related with spacetime transformations!

Another important consequence of (4.2.10) arises from the last commutator, which is $[P_\mu, Q_\alpha] = 0$. This implies that the states transformed by Q_α, namely $|\text{Boson}\rangle$ and $|\text{Fermion}\rangle$, have the same momentum and energy. Hence, because of $E^2 = \vec{p}^2 + m^2$, they also need to have the same mass, so $m_F = m_B$. This stops being true when one considers spontaneous SUSY breaking, whose formalism is not discussed here. It is also a key result that the number of bosonic and fermionic states coincide. We now proceed to prove this.

Proof. Let us consider a fermion number operator N_F, satisfying $N_F\,|\text{Fermion}\rangle = |\text{Fermion}\rangle$ and $N_F\,|\text{Boson}\rangle = 0$. This means that the operator $(-1)^{N_F}$ has eigenvalue $+1$ and -1 on bosonic and fermionic states respectively

$$(-)^{N_F}\,|\text{Boson}\rangle = +\,|\text{Boson}\rangle\,, \qquad (-)^{N_F}\,|\text{Fermion}\rangle = -\,|\text{Fermion}\rangle\,. \tag{4.2.13}$$

Then the operator $(-1)^{N_F}$ anticommutes with Q, $(-1)^{N_F}Q = -Q(-1)^{N_F}$, as we can check:

$$\begin{aligned}\left[(-1)^{N_F}Q + Q(-1)^{N_F}\right]|\text{Boson}\rangle &= (-1)^{N_F}\,|\text{Fermion}\rangle + Q\,|\text{Boson}\rangle \\ &= -\,|\text{Fermion}\rangle + |\text{Fermion}\rangle = 0.\end{aligned} \tag{4.2.14}$$

We now compute the following trace:

$$\begin{aligned}\text{Tr}\left[(-1)^{N_F}\left\{Q,\bar{Q}\right\}\right] &= \text{Tr}\left[(-1)^{N_F}\left(Q\bar{Q}+\bar{Q}Q\right)\right] \\ &= \text{Tr}\left[-Q(-1)^{N_F}\bar{Q} + Q(-1)^{N_F}\bar{Q}\right] = \text{Tr}[0] = 0.\end{aligned} \tag{4.2.15}$$

We have used the linear and cyclic properties of the trace, that is $\text{Tr}(A+B) = \text{Tr}A + \text{Tr}B$ and $\text{Tr}(AB) = \text{Tr}(BA)$, and also that $(-1)^{N_F}$ anticommutes with Q. But because of (4.2.10) we also know that $\left\{Q,\bar{Q}\right\} = \gamma_\mu P^\mu$, which means

$$\text{Tr}\left[(-1)^{N_F}\left\{Q,\bar{Q}\right\}\right] = \gamma_\mu \text{Tr}\left\{(-1)^{N_F}P^\mu\right\} = 0. \tag{4.2.16}$$

In order to avoid a zero momentum P_μ, we must have $\text{Tr}(-1)^{N_F} = 0$. This implies that there is an equal number of eigenvalues $+1$ and -1. In other words, there must be an equal number of bosonic and fermionic states. □

Therefore the single-particle states of a global SUSY theory can be grouped in multiplets, which contain equal number of bosons and fermions. The simplest multiplets are

1. The **chiral multiplet**, which contains a spin-1/2 fermion described by the Majorana field $\chi(x)$, plus its spin-0 bosonic partner, the sfermion, described by the complex scalar field $Z(x)$.

2. The **gauge multiplet**, which consists of a massless spin-1 particle, described by a vector gauge field $A_\mu(x)$, plus its spin-1/2 fermionic partner, the gaugino, described by a Majorana field $\lambda(x)$.

We are going to discuss two theories related to these multiplets in the following section. We end this part by simply stressing that there is an alternative approach to supersymmetry, called the superspace formalism, that we have not covered here.

4.3 Supersymmetric Lagrangians

4.3.1 The Wess-Zumino model

In this section, we are going to consider the Wess-Zumino model, which is the simplest example of interacting supersymmetric field theory. It is a chiral multiplet containing a complex scalar field $Z(x)$ and a Majorana field $\chi(x)$. For simplicity we are going to consider only its left-chiral projection $P_L\chi(x) = \boldsymbol{\chi}_L$, since from Section 2.3.2 we know the situation is equivalent.

In order to understand the form of the SUSY transformations of this chiral multiplet it is convenient to firstly discuss the dimensions of the quatities we are going to deal with. We are using natural units, so both c and $\hbar$ become adimensional numbers equal to 1. From this we can infer that the dimensions of time and length are the same and equal to dimensions of inverse of energy, i.e., $[T] = [L] = [E]^{-1}$. Taking into account that $E = mc^2$ becomes $E = m$, we see that energy and mass have the same dimensions, $[E] = [M]$. Everything is thus expressed in terms of powers of energy, $[E]^n$, so we will express the units in terms of n. Therefore we write:

$$[T] = [L] = -1, \qquad [M] = 1. \tag{4.3.1}$$

This means that the dimensions of a derivative is $[\partial_\mu] = 1$. The action $S = \int \mathrm{d}^4x\ \mathcal{L}$ is dimensionless, so the Lagrangian density must have dimension $[\mathcal{L}] = 4$. From this we can obtain the dimension of the fields. For example the kinetic term of a scalar field ϕ is given by $\partial_\mu\phi\partial^\mu\phi$, which means that a scalar field has dimensions $[\phi] = 1$. On the other hand, if we consider the mass term of a spinor field Ψ, which is $m\bar{\Psi}\Psi$, we see that spinors need to have dimension $[\Psi] = 3/2$.

Let us now introduce the infinitesimal supersymmetric transformations of the chiral multiplet:

$$\delta Z = \bar{\varepsilon} P_L \chi, \tag{4.3.2}$$

$$\delta(P_L\chi) = \sigma^\mu (P_R\varepsilon)\partial_\mu Z. \tag{4.3.3}$$

Equivalently, using that$P_L = P_L P_L$, we have

$$\delta Z = \bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L, \tag{4.3.4}$$

$$\delta \boldsymbol{\chi}_L = \sigma^\mu \boldsymbol{\varepsilon}_R \partial_\mu Z. \tag{4.3.5}$$

We are going to argue why (4.3.4) and (4.3.5) are actually the correct transformations. The first thing to notice is that the variation of Z gives rise to $\boldsymbol{\chi}_L$ whereas the variation of $\boldsymbol{\chi}_L$ involves Z. This was to be expected, because this is the whole idea of SUSY: bosons are transformed into fermions, and viceversa. Moreover, these transformations are linear in the fields. Non-linear transformations could have also be considered, but this would considerably complicate the computations.

Secondly, we can check that both sides of equations (4.3.4) and (4.3.5) transform in the same way under Lorentz transformations, as they should. For example, the contracted quantity $\bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L$ transforms as a Lorentz scalar, just as Z. On the other hand, $\boldsymbol{\chi}_L$ is a left-chiral spinor, so the quantity $\sigma^\mu \boldsymbol{\varepsilon}_R \partial_\mu Z$ should also transform as a left-chiral spinor. If we rewrite it as $\sigma^\mu \partial_\mu Z \boldsymbol{\varepsilon}_R$ and remember equations (2.2.27) we see that this does transform as a left-chiral spinor (since the fact that the derivative acts on the scalar field Z does not affect the behaviour under a Lorentz transformation).

Finally, let us check the dimensions. In order for equation (4.3.4) to have the correct dimensions, the infinitesimal SUSY parameter needs to have dimensions $[\boldsymbol{\varepsilon}_L] = -1/2$. This is consistent with the dimensions of (4.3.5), as $[\sigma^\mu \boldsymbol{\varepsilon}_R \partial_\mu Z] = -\frac{1}{2} + 1 + 1 = \frac{3}{2} = [\boldsymbol{\chi}_L]$.

Let us study now the action of the massless, non-interacting Wess-Zumino model:

$$S_{\text{kin}} = \int \mathrm{d}^4 x \ \left(\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}\right), \quad \text{where}$$

$$\mathcal{L}_{\text{scalar}} = -\partial^\mu Z^* \partial_\mu Z, \qquad \mathcal{L}_{\text{fermion}} = -\bar{\boldsymbol{\chi}} \not{\partial} P_L \boldsymbol{\chi} = -\bar{\boldsymbol{\chi}}_R \bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L. \tag{4.3.6}$$

We want to show that this simple action, containing only kinetic terms, is invariant under the transformations (4.3.4) and (4.3.5). After this, we will add interaction terms. For the present task we have to obtain the conjugate of the transformations (4.3.4) and (4.3.5). Using the results developed in Appendix D.2.4, we get:

$$\delta Z^* = \bar{\boldsymbol{\varepsilon}}_R \boldsymbol{\chi}_R, \tag{4.3.7}$$

$$\delta \bar{\boldsymbol{\chi}}_R = -\bar{\boldsymbol{\varepsilon}}_L \sigma^\mu \partial_\mu Z^*. \tag{4.3.8}$$

Let us now compute the infinitesimal transformations of $\mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{fermion}}$ separately. For the scalar part, we find

$$\delta \mathcal{L}_{\text{scalar}} = -\partial_\mu Z \partial^\mu (\delta Z)^* - \partial^\mu Z^* \partial_\mu (\delta Z) = -\bar{\boldsymbol{\varepsilon}}_R \partial^\mu \boldsymbol{\chi}_R \partial_\mu Z - \partial^\mu Z^* \bar{\boldsymbol{\varepsilon}}_L \partial_\mu \boldsymbol{\chi}_L. \tag{4.3.9}$$

For the fermion part, we obtain

$$\delta\mathcal{L}_{\text{fermion}} = -(\delta\bar{\boldsymbol{\chi}}_R)\bar{\sigma}^\mu\partial_\mu\boldsymbol{\chi}_L - \bar{\boldsymbol{\chi}}_R\bar{\sigma}^\mu\partial_\mu(\delta\boldsymbol{\chi}_L) = \bar{\boldsymbol{\varepsilon}}_L\sigma^\nu\partial_\nu Z^*\bar{\sigma}^\mu\partial_\mu\boldsymbol{\chi}_L - \bar{\boldsymbol{\chi}}_R\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu Z\boldsymbol{\varepsilon}_R. \tag{4.3.10}$$

We are going to show that $\delta\mathcal{L}_{\text{scalar}}$ precisely cancels $\delta\mathcal{L}_{\text{fermion}}$. We need to change the position of the derivatives in (4.3.10). With the help of the identity (2.1.6) we see that $\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu = \partial^\mu\partial_\mu$. Using this and integrating by parts, we can rewrite (4.3.10) as

$$\begin{aligned}\delta\mathcal{L}_{\text{fermion}} = {} & \partial^\mu Z^*\bar{\boldsymbol{\varepsilon}}_L\partial_\mu\boldsymbol{\chi}_L + \bar{\boldsymbol{\varepsilon}}_R\partial^\mu\boldsymbol{\chi}_R\partial_\mu Z + \\ & + \partial_\mu\left(\bar{\boldsymbol{\varepsilon}}_L\bar{\sigma}^\mu\sigma^\nu\boldsymbol{\chi}_L\partial_\nu Z^* - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\chi}_L\partial^\mu Z^* + \bar{\boldsymbol{\varepsilon}}_R\boldsymbol{\chi}_R\partial^\mu Z\right).\end{aligned} \tag{4.3.11}$$

The two first terms cancel exactly against $\delta\mathcal{L}_{\text{scalar}}$, and the remaining total derivative vanishes under the action integral after applying Gauss theorem. Therefore we have shown the invariance of the action under SUSY transformations

$$\delta S_{\text{kin}} = \int \mathrm{d}^4x \;\left(\delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}}\right) = 0. \tag{4.3.12}$$

But we still have not finished showing that the action (4.3.6) is supersymmetric. We need to check that the algebra of transformations agrees with (4.2.10). A way of doing so is taking the field $\hat{Z}$ and computing the commutator of two succesive SUSY variations. Using (4.3.4) and (4.3.5), we find

$$\begin{aligned}[\delta_1,\delta_2]\hat{Z} &= \delta_1(\bar{\boldsymbol{\varepsilon}}^2_L\boldsymbol{\chi}_L) - \delta_2(\bar{\boldsymbol{\varepsilon}}^1_L\boldsymbol{\chi}_L) = \bar{\boldsymbol{\varepsilon}}^2_L\sigma^\mu\boldsymbol{\varepsilon}^1_R\partial_\mu\hat{Z} - \bar{\boldsymbol{\varepsilon}}^1_L\sigma^\mu\boldsymbol{\varepsilon}^2_R\partial_\mu\hat{Z} \\ &= -\left[\bar{\boldsymbol{\varepsilon}}^1_R\bar{\sigma}^\mu\boldsymbol{\varepsilon}^2_L + \bar{\boldsymbol{\varepsilon}}^1_L\sigma^\mu\boldsymbol{\varepsilon}^2_R\right]\partial_\mu\hat{Z} = -\bar{\varepsilon}_1\gamma^\mu\varepsilon_2\partial_\mu\hat{Z},\end{aligned} \tag{4.3.13}$$

where we have used the identity $\bar{\boldsymbol{\varepsilon}}^2_L\sigma^\mu\boldsymbol{\varepsilon}^1_R = -\bar{\boldsymbol{\varepsilon}}^1_R\bar{\sigma}^\mu\boldsymbol{\varepsilon}^2_L$ and the Weyl representation (2.2.5) for γ^μ. This equation agrees with (4.2.12), as we expected. But notice that we should obtain the same result for the field $\hat{\boldsymbol{\chi}}_L$. If one computes $[\delta_1,\delta_2]\boldsymbol{\chi}_L$, it is found that it only agrees with (4.2.12) if the equations of motion arising from the action (4.3.6) are used. It is then said that the superalgebra only closes *on-shell* (this means that equations of motions are satisfied). For many reasons (see [17]), it is normally wished that the superalgebra closes *off-shell* (that is, for arbitrary field configurations, without imposing equations of motion). In order to solve this problem, one needs to add what is called an **auxiliary field** F. This is a complex scalar field that carries no dynamics and simply helps in the intermediate steps. It is added to the kinetic action S_{kin} in the following manner

$$S_{\text{kin}} = \int \mathrm{d}^4x \;\left(\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}} + FF^*\right). \tag{4.3.14}$$

In this way, the equations of motion describing F are algebraic (that is to say, they do not contain derivatives) so F can be eliminated at a later stage. Notice that the dimension of F is $[F] = 2$, unlike Z. As we want to take advantage of the supersymmetric transformation of F for making the superalgebra close off-shell, one possibility is to take δF as a multiple of the equations of motion (which are $\partial^\mu \partial_\mu Z = 0$ and $\bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L = 0$). If we consider the infinitesimal transformation

$$\delta F = \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L, \tag{4.3.15}$$

then the transformation (4.3.5) for $\boldsymbol{\chi}_L$ needs to be modified in order to maintain the invariance of S_{kin}:

$$\delta \boldsymbol{\chi}_L = \sigma^\mu \boldsymbol{\varepsilon}_R \partial_\mu Z + \boldsymbol{\varepsilon}_L F. \tag{4.3.16}$$

So far, we have shown a theory of free fields that is invariant under supersymmetric transformations. But if we are looking for a realistic model, we need to introduce interaction terms among the fields in the Lagrangian, and this is what we are going to do now. The interaction action S_{int} that we seek needs to have the following general form [8]

$$S_{\text{int}} = \int \mathrm{d}^4 x \, \left(W_1 F - \frac{1}{2} W_{11} \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L + W_1^* F^* - \frac{1}{2} W_{11}^* \bar{\boldsymbol{\chi}}_R \boldsymbol{\chi}_R \right), \tag{4.3.17}$$

where $W_1 = W_1(Z, Z^*)$ and $W_{11} = W_{11}(Z, Z^*)$ are arbitrary functions of the scalar field. Notice that the presence of the conjugate of $W_1 F - \frac{1}{2} W_{11} \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L$ is justified to make the action real. We are going to find, by imposing invariance under supersymmetry transformations, that the yet unspecified functions W_1 and W_{11} are related.

For this task, we will call $\mathcal{L}_1 = W_1 F$ and $\mathcal{L}_2 = -\frac{1}{2} W_{11} \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L$. Our aim is to show that $\delta \mathcal{L}_1$ and $\delta \mathcal{L}_2$ cancel against each other (leaving at most a total derivative term). We will not need to compute the conjugate terms; if the terms of $\delta \mathcal{L}_1$ and $\delta \mathcal{L}_2$ cancel out, their conjugates will cancel out as well. We first compute $\delta \mathcal{L}_1$:

$$\begin{aligned} \delta \mathcal{L}_1 = \delta W_1 F + W_1 \delta F &= \left(\frac{\partial W_1}{\partial Z} \delta Z + \frac{\partial W_1}{\partial Z^*} \delta Z^* \right) F + W_1 \delta F \\ &= \underbrace{\frac{\partial W_1}{\partial Z} \bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L F}_{\text{I}} + \underbrace{\frac{\partial W_1}{\partial Z^*} \bar{\boldsymbol{\varepsilon}}_R \boldsymbol{\chi}_R F}_{\text{II}} + \underbrace{W_1 \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L}_{\text{III}} . \end{aligned} \tag{4.3.18}$$

For $\delta \mathcal{L}_2$, we obtain:

$$\begin{aligned} \delta \mathcal{L}_2 &= -\frac{1}{2} \delta W_{11} \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L - W_{11} \bar{\boldsymbol{\chi}}_L \delta \boldsymbol{\chi}_L = -\frac{1}{2} \left(\frac{\partial W_{11}}{\partial Z} \delta Z + \frac{\partial W_{11}}{\partial Z^*} \delta Z^* \right) \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L - W_{11} \bar{\boldsymbol{\chi}}_L \delta \boldsymbol{\chi}_L \\ &= \underbrace{-\frac{1}{2} \frac{\partial W_{11}}{\partial Z} \bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L}_{\text{IV}} \underbrace{- \frac{1}{2} \frac{\partial W_{11}}{\partial Z^*} \bar{\boldsymbol{\varepsilon}}_R \boldsymbol{\chi}_R \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L}_{\text{V}} - \underbrace{W_{11} \bar{\boldsymbol{\chi}}_L \sigma^\mu \boldsymbol{\varepsilon}_R \partial_\mu Z}_{\text{VI}} - \underbrace{W_{11} \bar{\boldsymbol{\chi}}_L \boldsymbol{\varepsilon}_L F}_{\text{VII}} . \end{aligned} \tag{4.3.19}$$

[8] The reason for having this form is that this is the most general interaction action that can lead to a renormalizable theory when quantized [17]. The factor $-\frac{1}{2}$ is added just for simplicity in the calculations.

We have labeled the terms with Roman numbers in order to make it easier to refer to them. Let us discuss each term separately. The term IV contains $\bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L$, which is identically zero because it contains the square of a Grassmann anticommuting number. The term V contains $\bar{\boldsymbol{\varepsilon}}_R \boldsymbol{\chi}_R \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L$, which is not zero, but it cannot cancel with any other term, so we must impose

$$\frac{\partial W_{11}}{\partial Z^*} = 0. \tag{4.3.20}$$

This means that the complex function W_{11} does not depend on Z^*, that is to say, W_{11} is a *holomorphic* function of Z. In the same way, if we have a look at the term II, we see there are no other terms that could possibly cancel against it, so we need to impose again that

$$\frac{\partial W_1}{\partial Z^*} = 0. \tag{4.3.21}$$

Thus W_1 is also a holomorphic function of Z. We are still left with four terms. We group the two terms containing derivatives to see if they cancel out

$$\begin{aligned} \text{III} + \text{VI} &= W_1 \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L - W_{11} \bar{\boldsymbol{\chi}}_L \sigma^\mu \boldsymbol{\varepsilon}_R \partial_\mu Z \\ &= W_1 \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L + W_{11} \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \boldsymbol{\chi}_L \partial_\mu Z. \end{aligned} \tag{4.3.22}$$

We proceed to integrate by parts the first term. We will obtain a total derivative term which vanishes under the action integral. We get

$$\text{III} + \text{VI} = -\partial_\mu W_1 \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \boldsymbol{\chi}_L + W_{11} \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \boldsymbol{\chi}_L \partial_\mu Z + \partial_\mu \left(W_1 \bar{\boldsymbol{\varepsilon}}_R \bar{\sigma}^\mu \boldsymbol{\chi}_L \right). \tag{4.3.23}$$

The first two terms cancel out provided that we impose

$$W_{11} \partial_\mu Z = \partial_\mu W_1. \tag{4.3.24}$$

This is the relation between W_{11} and W_1 we mentioned earlier. Applying the chain rule we have $\partial_\mu W_1 = \frac{\partial W_1}{\partial Z} \partial_\mu Z$, so the condition (4.3.24) can alternatively be expressed as

$$\boxed{\frac{\partial W_1}{\partial Z} = W_{11}}. \tag{4.3.25}$$

Regarding the remaining two terms

$$\text{I} + \text{VII} = \frac{\partial W_1}{\partial Z} \bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L F - W_{11} \bar{\boldsymbol{\chi}}_L \boldsymbol{\varepsilon}_L F = \left(\frac{\partial W_1}{\partial Z} - W_{11} \right) \bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L F = 0, \tag{4.3.26}$$

where we have used $\bar{\boldsymbol{\varepsilon}}_L \boldsymbol{\chi}_L = \bar{\boldsymbol{\chi}}_L \boldsymbol{\varepsilon}_L$. Therefore we have finally shown that the kinetic and the interaction actions are independently invariant,

$$\delta S_{\text{kin}} = 0, \qquad \delta S_{\text{int}} = 0, \tag{4.3.27}$$

under the supersymmetric transformations (4.3.4), (4.3.15) and (4.3.16). It is often convenient to introduce a new function of Z related to W_1 by

$$\boxed{W_1 \equiv \frac{\partial \mathcal{W}}{\partial Z}}. \tag{4.3.28}$$

$\mathcal{W}(Z)$ is called *the superpotential*. We write in detail the total supersymmetric action:

$$\begin{aligned} S = S_{\text{kin}} + S_{\text{int}} &= \int \mathrm{d}^4x \; \mathcal{L}_{WZ} \\ &= \int \mathrm{d}^4x \left(-\partial^\mu Z^* \partial_\mu Z - \bar{\boldsymbol{\chi}}_R \bar{\sigma}^\mu \partial_\mu \boldsymbol{\chi}_L + FF^* + \frac{\partial \mathcal{W}}{\partial Z} F - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial Z^2} \bar{\boldsymbol{\chi}}_L \boldsymbol{\chi}_L + \text{cc} \right), \end{aligned} \tag{4.3.29}$$

where cc denote the conjugate terms. $\mathcal{L}_{WZ}$ is known as the *interacting Wess-Zumino* Lagrangian density. In particular, Wess and Zumino considered the following superpotential (see [5]):

$$\mathcal{W}(Z) = \frac{1}{2} m Z^2 + \frac{1}{6} g Z^3, \tag{4.3.30}$$

The first part gives rise to the mass terms whereas the second part is the coupling. The equation of motion for the auxiliary field is simply $F = -\partial W^* / \partial Z$, which can be substituted back in (4.3.29) in order to eliminate F and F^*.

4.3.2 SUSY Yang-Mills theory

SUSY Yang-Mills theory is another interacting supersymmetric theory, in which the inclusion of a gauge multiplet is considered. It contains the gauge vector fieds $A_\mu^A(x)$ and its superpartners, the gauginos λ^A, which are massless Majorana spinors. The gauginos λ^A transform in the adjoint representation of a non-Abelian group G (we have discussed the adjoint representation in Appendix C.2). The action is given by

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4} F^{\mu\nu A} F^A_{\mu\nu} - \frac{1}{2} \bar{\lambda}^A \gamma^\mu D_\mu \lambda^A \right), \tag{4.3.31}$$

where $D_\mu \lambda^A = \partial_\mu \lambda^A + g f_{BCA} A^B_\mu \lambda^C$. The equations of motion are

$$D^\mu F^A_{\mu\nu} = -\frac{1}{2} g f_{BCA} \bar{\lambda}^B \gamma_\nu \lambda^C, \tag{4.3.32}$$

$$\gamma^\mu D_\mu \lambda^A = 0, \tag{4.3.33}$$

while the gauge fields also satisfy the Bianchi identity:

$$D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0. \tag{4.3.34}$$

In this case, we follow a different approach from what we did in the Wess-Zumino model. The idea is to make use of Noether's theorem (see Appendix B.1.2). If an action is invariant under a certain continuous transformation, Noether's theorem tells us that there exists an associated conserved current.

The conserved current related with a supersymmetric transformation must have a spinor nature, since its spatial integral needs to give rise to a supercharge. Thus it is a vector-spinor object, called *supercurrent* $\mathcal{J}^\mu_\alpha$. The procedure consists of finding a supercurrent that is conserved when equations of motion are imposed. Then the supercharge can be computed as

$$Q_\alpha = \int \mathrm{d}^3x \; \mathcal{J}^0_\alpha(\vec{x}, t). \tag{4.3.35}$$

After this, making use of (4.2.11) one can obtain the SUSY transformations that leave the action invariant.

Let us proceed in this way for SUSY Yang-Mills theory. The supercurrent, omitting spinorial indices, is given by

$$\boxed{\mathcal{J}^\mu = \gamma^{\nu\rho} F^A_{\nu\rho} \gamma^\mu \lambda^A}. \tag{4.3.36}$$

We are going to prove that it is conserved.

***Proof*.** The supercurrent in (4.3.36) has only one free index, μ (and a spinor index α in λ^A_α that we do not write). Thus, because the current will be a scalar under the Yang-Mills group, $\partial_\mu \mathcal{J}^\mu = D_\mu \mathcal{J}^\mu$. Taking advantage of this fact, we compute

$$\begin{aligned}
\partial_\mu \mathcal{J}^\mu &= D_\mu \mathcal{J}^\mu = D_\mu F^A_{\nu\rho} \gamma^{\nu\rho} \gamma^\mu \lambda^A + \gamma^{\nu\rho} F^A_{\nu\rho} \gamma^\mu D_\mu \lambda^A \\
&= D_\mu F^A_{\nu\rho} \gamma^{\nu\rho} \gamma^\mu \lambda^A = -2 D^\mu F^A_{\mu\nu} \gamma^\nu \lambda^A \\
&= g f_{ABC} \gamma^\nu \lambda^A \bar{\lambda}^B \gamma_\nu \lambda^C = g f_{ABC} \gamma^\nu \lambda^{[A} \bar{\lambda}^B \gamma_\nu \lambda^{c]} = 0.
\end{aligned} \tag{4.3.37}$$

Note that in the second line we have used the Bianchi identities (4.3.34) and the relation $\gamma^{\nu\rho} = \gamma^\nu \gamma^\rho - \eta^{\nu\rho} \mathbb{1}$. The term in the last line vanishes due tobecause of the Fierz identity (D.2.21) we have proved in Appendix D.2.3. □

Therefore we have shown that the action (4.3.31) is supersymmetric.

Conclusions

The goal of this thesis was to analyse two fundamental SUSY theories: the Wess-Zumino model and the supersymmetric Yang-Mills theory. After an extensive study of the fundamental concepts of bosonic and fermionic fields, we have been able to achieve this goal. We have seen that SUSY provides a natural extension to the symmetries of the SM that is mathematically self-consistent. We have also seen that this new formulation offers many advantages, such as the unification of internal and spacetime symmetries or solutions to the many open problems in particle physics.

In Chapter 1 we have studied the Klein-Gordon field for two reasons. The first one is that it describes spin-0 bosons, which certainly plays a role in SUSY theories. The second one is that, given its simplicity, it is a feasible scenario to investigate the conserved quantities through the Noether formalism.

In Chapter 2 we have studied the basic concepts concerning spinors, in order to properly describe spin-$\frac{1}{2}$ fermions. We have showed the deep relation between spin and Lorentz symmetry, and we have understood the important concept of the spinorial representations of the Lorentz group. We have studied three types of spinors: Dirac, Weyl and Majorana spinors; and we have seen that the existence of the last two is restricted for certain dimensions. The concept of Majorana spinor has been proved to be essential for developing SUSY.

In Chapter 3 we have investigated gauge fields, which are needed to describe spin-1 bosons. We have seen that these theories describe interactions between particles, that arise from imposing invariance under local transformations. We have studied the Maxwell field, based on an Abelian $U(1)$ gauge symmetry, and later the Yang-Mills fields, which are a generalization of electromagnetism that considers non-Abelian gauge symmetries.

In Chapter 4 we have finally studied SUSY, which was the main objective of this thesis. We have seen that SUSY offers the only loophole to the Coleman-Mandula theorem, which forbids spacetime and internal symmetries to be blended. We have studied the Wess-Zumino model and SUSY Yang-Mills theory, which are able to describe renormalizable interactions. We have also enumerated several motivating reasons for studying SUSY.

Throughout the different appendices, we have developed all the necessary tools for this work. For instance, we have seen the main aspects of Lagrangian and canonical formalism, we have learned the basics of Lie groups (with emphasis in the Lorentz group) and we have investigated some basic notions of Clifford algebras and their application to spinors in arbitrary dimension.

The methodology has consisted of analytical calculations by hand, and the use of the Mathematica software when necessary. In order to acquire a solid understanding of the subject, around fifty exercises proposed in the first 5 chapters of [30] have been done by the author. For finding bibliographical sources we have made use of the open access repository arXiv.

Let us comment now some prospects of this work. It is premature to conclude wether the Large Hadron Collider will detect SUSY particles or not, since it has recently started to run at a doubled energy (14 TeV in total) and there are still hundreds of data to collect. The precise values of the superpartners masses are not known, but they may sit at a few TeV, which is perfectly accessible by the LHC [31]. What is clear is that finding SUSY would mean a revolution in the current understanding of Nature. This would not only deserve a Nobel price but it would also mean a strong support for unified theories such as supergravity or superstrings.

The fact is that even not finding SUSY would have strong implications: the MSSM is the only model that currently solves the three problems of naturalness, gauge coupling unification and dark matter all at once, so ruling it out will force to consider other better alternatives. It is also worth saying that SUSY has worked until now as an ideal toy model for theorists and has helped to develop new proposals. For example, $\mathcal{N} = 4$ SUSY Yang-Mills has lead to the most succesful realization of the holographic principle. In any case, it may take long until we certainly know if SUSY becomes a physical fact, in the same way it took a century to discover the gravitational waves proposed by Albert Einstein.

We would like to finish saying that this work has had a positive impact on the educational background of the author, since he has learned some tools that are used by theoretical physicists in a daily basis. For instance, he has become comfortable with group theory and he has learned tensorial calculus with skill. Ultimately, he has understood how to work in arbitrary dimension and the advantages of it. We interpret this work as an introductory project on supersymmetry which will open the door to futures researches in theoretical physics.

Appendix A

Conventions

In this appendix we establish the conventions that have been followed doing this document.

- Natural units for which $\hbar = \varepsilon_0 = c = 1$ are used throughout the work.
- We consider the D-dimensional Minkowski metric according to the "mostly-plus" signature $\eta_{\mu\nu} = \text{diag}(-, +, ..., +)$. It has $D - 1$ spatial dimensions and 1 time dimension.
- We denote Cartesian coordinates by x^μ, $\mu = 0, 1, ..., D-1$ with time coordinate $x^0 = t$. We use relativistic notation, so spacetime coordinates are labeled with greek indices. We often write fields as $\phi(x)$, which is a shorthand notation for $\phi(x^0, x^1, ..., x^{D-1})$.
- Einstein summation convention is assumed, so summations for dummy indices are removed. For example, we write $x^\nu = \eta^{\nu\mu} x_\mu$.
- We denote Lie algebras in the same way as the corresponding group G, but using gothic letters, $\mathfrak{g}$.
- Matrices are multiplied with dummy indices in up-down position. For example, $(AB)^\mu{}_\nu = A^\mu{}_\rho B^\rho{}_\nu$. Identity matrix is always written as $\mathbb{1}$, no matter its dimension.
- We use square brackets to emphasize the antisymmetry of two indices. For example $\gamma^{[\mu\nu]} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. Round brackets () denote symmetry of indices.
- We use arrows for the spatial components of vectors, so we write $x^\mu = (t, \vec{x})$. Scalar product are written as $A \cdot B = \eta^{\alpha\beta} A_\alpha B_\beta$ $(\alpha = 0, ..., D-1)$. Arrows are only used for the scalar product of the spatial parts, for example $\vec{A} \cdot \vec{B} = A^i B_i$ $(\ i = 1, ..., D-1)$.

Appendix B

The alphabet of Classical Field Theory

In this appendix we develop some basic tools of classical field theory. The results exposed here are valid for any spacetime dimension D.

In the context of physics, a field ϕ (or a set of fields $\{\phi^i\}_{i=1,\ldots,N}$) is a function (or a set of functions) exhibiting a dependence on space and time. Depending on the Lorentz representation under which the fields transform, they can be classified as **scalars**, **spinors**, **vectors** or **tensors**. A field can also be regarded as *classical* or *quantum*. The former is described by complex or real numbers, whereas the later is represented by an operator in Hilbert space. Here we address classical fields.

The fields are assumed to take values over a D-dimensional flat spacetime, described by the metric tensor $\eta_{\mu\nu}$. The metric tensor $\eta_{\mu\nu}$ is used to lower vector indices whereas the inverse metric $\eta^{\mu\nu}$ raises them. Upper and lower indices denote contravariant and covariant tensors, respectively.

B.1 Lagrangian formalism

From Lagrangian mechanics, we know that all the dynamical information about the fields ϕ^i is contained in the Lagrangian density $\mathcal{L}$, which generally depends on the field and its first derivatives

$$\mathcal{L}(x, \phi^i, \partial_\mu \phi^i), \tag{B.1.1}$$

being $\partial_\mu \phi^i = \frac{\partial \phi^i}{\partial x^\mu}$. **The action** S is a functional, a real number that depends on the configurations of the fields. It is given by

$$S[\phi^i] = \int_\Omega \mathrm{d}^D x \; \mathcal{L}(x, \phi^i, \partial_\mu \phi^i), \tag{B.1.2}$$

where Ω is a spacetime region and $\mathrm{d}^D x = \mathrm{d}x^0 \mathrm{d}x^1 \cdots \mathrm{d}x^{D-1}$ is the volume element in Cartesian coordinates. The frontier of Ω is denoted by $\partial\Omega$. An infinitesimal variation of the fields, which can be expressed in terms of a parameter ε by $\delta\phi^i = \varepsilon\Delta\phi^i$, induces an infinitesimal variation on the action, δS. This can be defined as

$$\delta S = \int_\Omega \mathrm{d}^D x \, \frac{\delta S}{\delta \phi^i} \Delta\phi^i = \lim_{\varepsilon\to 0} \frac{S\left[\phi^i(x) + \varepsilon\Delta\phi^i\right] - S\left[\phi^i(x)\right]}{\varepsilon}. \tag{B.1.3}$$

Here $\delta S/\delta\phi^i$ is the so-called functional derivative of the action (following the approach given in [32])

B.1.1 Euler-Lagrange equations

The **principle of least action** is a variational principle used for obtaining the equations of motion of a system. For field theory, it states that the action S must be stationary, $\delta S = 0$, under an arbitrary variation $\delta\phi^i$ of the fields. We are going to use this in order to derive the Euler-Lagrange equations

$$\begin{aligned}
\delta S &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega \mathrm{d}^D x \, \left\{ \mathcal{L}\left(x, \phi^i(x) + \varepsilon\Delta\phi^i, \partial_\mu\phi^i(x) + \varepsilon\partial_\mu\Delta\phi^i\right) - \mathcal{L}\left(x, \phi^i(x), \partial_\mu\phi^i(x)\right)\right\} \\
&= \int_\Omega \mathrm{d}^D x \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left\{ \varepsilon \frac{\partial\mathcal{L}}{\partial\phi^i}\Delta\phi^i + \varepsilon \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\partial_\mu\Delta\phi^i + O(\varepsilon^2)\right\} \\
&= \int_\Omega \mathrm{d}^D x \left\{ \frac{\partial\mathcal{L}}{\partial\phi^i}\Delta\phi^i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\Delta\phi^i\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\right)\Delta\phi^i\right\} \\
&= \int_\Omega \mathrm{d}^D x \left\{ \frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\right)\right\}\Delta\phi^i + \int_{\partial\Omega} \mathrm{d}^{D-1} x \, \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\Delta\phi^i = 0.
\end{aligned} \tag{B.1.4}$$

In the third line, integration by parts has been used, and in the last line, the term with the total derivative has become a boundary term after applying Gauss Theorem. The principle of least action requires that the fields are fixed at the boundary, so $\Delta\phi^i = 0$ in $\partial\Omega$. Finally, taking into account that $\Delta\phi^i$ are independent and that the region Ω is arbitrary, we arrive at

$$\boxed{\frac{\delta S}{\delta\phi^i} = \frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\right) = 0}. \tag{B.1.5}$$

These are the so-called Euler-Lagrange equations, which govern the dynamical evolution of the system.

B.1.2 Noether's Theorem

We say that, for fields $\phi^i(x)$ satisfying the equations of motion (B.1.5), a transformation $\phi^i(x) \to \phi'^i(x)$ is a **symmetry** if $\phi'^i(x)$ also satisfies the equations of motion. This transformation can correspond to a spacetime symmetry, which affects the coordinates x, such as translations $\phi'^i(x) = \phi^i(x+a)$, or an internal symmetry, like rotations in the internal space of the fields $\phi'^i(x) = R^i{}_j\phi^j(x)$. For most of the systems we will study, symmetry transformations simply leave the action invariant

$$S[\phi'^i] = S[\phi^i]. \tag{B.1.6}$$

Noether's theorem relates continuous symmetries with conservation laws and it is one of the major results in theoretical physics [33]. We proceed to derive it. Firstly, we extend the previous definition of infinitesimal variation of the fields to the case where there are many independent transformation parameters, labeled by $A = 1, ..., p$:

$$\delta\phi^i(x) \equiv \epsilon^A \Delta_A \phi^i(x). \tag{B.1.7}$$

This formula includes the two different cases of spacetime and internal symmetries [1]. We are going to impose that (B.1.7) is a symmetry of the theory, i.e. that it satisfies (B.1.6). Then, the transformed and the original Lagrangian densities differ by a total derivative $\delta\mathcal{L} = \varepsilon^A \partial_\mu K^\mu_A$. This leads to a boundary term (after using Gauss Theorem), which can be set to zero because of the assumption that the fields ϕ^i vanish at large distances. The variation of $\mathcal{L}$ is

$$\begin{aligned}\delta\mathcal{L} =&\mathcal{L}(x, \phi^i + \varepsilon^A\Delta_A\phi^i, \partial_\mu\phi^i + \varepsilon^A\partial_\mu(\Delta_A\phi^i)) - \mathcal{L}(x, \phi^i, \partial_\mu\phi^i) \\ =&\varepsilon^A\left[\frac{\partial\mathcal{L}}{\partial\phi^i}\Delta_A\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\partial_\mu(\Delta_A\phi^i)\right] = \varepsilon^A\partial_\mu K^\mu_A.\end{aligned} \tag{B.1.8}$$

Now, by using the Euler-Lagrange equations (B.1.5):

$$\cancel{\frac{\partial\mathcal{L}}{\partial\phi^i}\Delta_A\phi^i} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\Delta_A\phi^i\right) - \cancel{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\right)}\Delta_A\phi^i = \partial_\mu K^\mu_A. \tag{B.1.9}$$

Therefore we can read the following continuity equation:

$$\partial_\mu J^\mu_A = 0, \tag{B.1.10}$$

where the conserved quantities J^μ_A are called **Noether currents**, given by

$$J^\mu_A = K^\mu_A - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\Delta_A\phi^i. \tag{B.1.11}$$

For each conserved current one can define a **Noether charge**,

$$Q_A = \int \mathrm{d}^{D-1}\vec{x}\ J^0_A(\vec{x}, t). \tag{B.1.12}$$

which is a constant of motion (that is, independent of time), provided that fields are damped at large distances

$$\partial_0 Q_A = \int \mathrm{d}^{D-1}\vec{x}\ \partial_0 J^0_A = -\int \mathrm{d}^{D-1}\vec{x}\ \partial_i J^i_A \to 0. \tag{B.1.13}$$

[1] It is also said that a spacetime symmetry is non-local, because it depends on the point x, whereas an internal symmetry is local.

We have made use of the continuity equation, $\partial_\mu J^\mu_A = \partial_0 J^0_A + \partial_i J^i_A = 0$ and of Gauss Theorem for converting the $(D-1)$-dimensional divergence $\partial_i J^i_A$ into a boundary integral. Noether currents are not unique. One can add terms like:

$$J'^\mu_A = J^\mu_A + \Delta J^\mu_A = J^\mu_A + \partial_\rho S^{\rho\mu}{}_A, \tag{B.1.14}$$

where $S^{\rho\mu}{}_A$ is any arbitrary antysimmetric function, $S^{\rho\mu}{}_A = -S^{\mu\rho}{}_A$, since:

$$\partial_\mu J'^\mu_A = \underbrace{\partial_\mu J^\mu_A}_{=0} + \partial_\mu \left(\partial_\rho S^{\rho\mu}{}_A\right) = \partial_\mu \partial_\rho S^{\rho\mu}{}_A = 0. \tag{B.1.15}$$

The term $\partial_\mu \partial_\rho S^{\rho\mu}{}_A$ is zero because it is a contraction of a symmetric part $\partial_\mu \partial_\rho$ with an antisymmetric one $S^{\rho\mu}{}_A$.

B.2 Canonical formalism

The canonical formalism is an alternative formulation in classical field theory, which describes systems by a set of canonical coordinates and momenta, forming the so-called **phase space**. One of the advantages that it offers is that it can be very easily generalized to the quantum theory. Here we just sketch some results of interest. At a fixed time $t = 0$, the canonical fields ϕ^i and the canonical momenta $\pi_i(\vec{x}, 0)$ are given by

$$\phi^i = \phi^i(\vec{x}, 0), \qquad \pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^i(\vec{x}, 0))}. \tag{B.2.1}$$

The Hamiltonian H of the system is obtained by integrating the Hamiltonian density $\mathcal{H}$, which is a Legendre transformation of $\mathcal{L}$

$$H = \int \mathrm{d}^{D-1}\vec{x}\ \mathcal{H}(x, \phi^i, \pi_i) = \int \mathrm{d}^{D-1}\vec{x} \left(\pi_i \partial_t \phi^i - \mathcal{L}\right). \tag{B.2.2}$$

From H one can obtain the equations of motion using the so-called Hamilton equations. However, for us it is more important the fact that H is a conserved quantity emerging from time translation symmetry.

We consider special cases of symmetries in which the time component of the vector K^μ_A in (B.1.11)is zero. Therefore, the formula (B.1.12) for the Noether charges reduces to

$$Q_A = -\int \mathrm{d}^{D-1}\vec{x}\ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^i)} \Delta_A \phi^i = -\int \mathrm{d}^{D-1}\vec{x}\ \pi_i \Delta_A \phi^i. \tag{B.2.3}$$

The Poisson bracket of any two observables $A(\phi, \pi)$ and $B(\phi, \pi)$ at two different points x and y is defined

$$\{A(\vec{x}), B(\vec{y})\}_P \equiv \int \mathrm{d}^{D-1}\vec{s} \left(\frac{\delta A(\vec{x})}{\delta \phi^i(\vec{s})} \frac{\delta B(\vec{y})}{\delta \pi_i(\vec{s})} - \frac{\delta A(\vec{x})}{\delta \pi_i(\vec{s})} \frac{\delta B(\vec{y})}{\delta \phi^i(\vec{s})}\right), \tag{B.2.4}$$

where a functional derivative is used $\frac{\delta A}{\delta \phi} = \frac{\partial A}{\partial \phi} - \partial_\mu \left(\frac{\partial A}{\partial(\partial_\mu \phi)} \right)$. One of the basic basic Poisson brackets is

$$\left\{ \phi^i(\vec{x}), \pi_j(\vec{y}) \right\}_P = \int \mathrm{d}^{D-1}\vec{s} \left(\frac{\delta \phi^i(\vec{x})}{\delta \phi^k(\vec{s})} \frac{\delta \pi_j(\vec{y})}{\delta \pi_k(\vec{s})} - \cancel{\frac{\delta \phi^i(\vec{x})}{\delta \pi_k(\vec{s})} \frac{\delta \pi_j(\vec{y})}{\delta \phi^k(\vec{s})}} \right)$$

$$= \int \mathrm{d}^{D-1}\vec{s}\ \underbrace{\frac{\partial \phi^i}{\partial \phi^k}}_{\delta^i_k} \delta(\vec{x} - \vec{s}) \underbrace{\frac{\partial \pi_j}{\partial \pi_k}}_{\delta^j_k} \delta(\vec{y} - \vec{s}) = \delta^i_j \delta^{D-1}(\vec{x} - \vec{y}). \tag{B.2.5}$$

An important result is that the infinitesimal symmetry transformation of a field $\Delta_A \phi^i$ is obtained by its Poisson bracket with the Noether charge Q_A:

$$\boxed{\Delta_A \phi^i(x) = \left\{ Q_A, \phi^i(x) \right\}_P}. \tag{B.2.6}$$

Proof.

$$\begin{aligned}
\left\{ Q_A, \phi^i(\vec{x}) \right\}_P &= -\int \mathrm{d}^{D-1}\vec{y}\ \left\{ \pi_j(\vec{y}) \Delta \phi^j(\vec{y}), \phi^i(\vec{x}) \right\}_P \\
&= -\int \mathrm{d}^{D-1}\vec{y}\ \left(\left\{ \pi_j(\vec{y}), \phi^i(\vec{x}) \right\}_P \Delta_A \phi^j(\vec{y}) + \cancel{\left\{ \Delta_A \phi^j(\vec{y}), \phi^i(\vec{x}) \right\}_P} \pi_j(\vec{y}) \right) \\
&= \int \mathrm{d}^{D-1}\vec{y}\ \delta^i_j \delta^{D-1}(\vec{x} - \vec{y}) \Delta_A \phi^j(\vec{y}) = \Delta_A \phi^i(\vec{x}).
\end{aligned} \tag{B.2.7}$$

□

The result (B.2.6) is also valid for time translations, in which case takes the form

$$\Delta_A \phi^i(x) = \partial_t \phi^i(x) = \left\{ H, \phi^i(x) \right\}_P, \tag{B.2.8}$$

but the computations needed to show this are a little bit more tedious (for the details, see [34]). It is worth mentioning that the Poisson brackets of the Noether charges obey the Lie algebra of the symmetry group

$$\{Q_a, Q_b\}_P = f_{abc} Q_c. \tag{B.2.9}$$

Here f_{abc} denote the structure constants of the group. Thus, since Noether charges generate infinitesimal transformations and contain the information about the f_{abc}, they provide a representation of the generators of the symmetry group.

These results can be generalized without many difficulties to the quantum case. In the quantum theory, for each classical observable A there is a corresponding operator in Hilbert space (distinguished with a hat, $\hat{A}$), and the Poisson bracket becomes a commutator. For instance

$$\{A, B\}_P = C \ \rightarrow \ [\hat{A}, \hat{B}] = i\hat{C}, \tag{B.2.10}$$

using that $\hbar = 1$. With this recipe, called canonical quantization, it is possible to obtain the quantum versions of the classical fields.

Appendix C

Basic notions of group theory

Group theory is a fundamental tool in theoretical physics, as it is deeply related to the notion of symmetry. A quotation from Sir Arthur Stanley Eddington perfectly summarises the importance of Group theory [35]:

> *"We need a super-mathematics in which the operations are as unknown as the quantities they operate on, and a super-mathematician who does not know what he is doing when he performs these operations. Such a super-mathematics is the Theory of Groups".*

In this appendix we review the most important concepts of group theory [36], making emphasis in the theory of Lie groups, and specially, in the Lorentz group.

C.1 Basic definitions

We firstly give some basic definitions

Definition C.1.1. *A **group**, G, is a set with a rule for assigning to every (ordered) pair of elements* [1]*, a third element, satisfying:*

1. *If $f, g \in G$ then $h = fg \in G$.*
2. *For $f, g, h \in G$, $f(gh) = (fg)h$.*
3. *There is an identity element, e, such that for all $f \in G$, $ef = fe = f$.*
4. *Every element $f \in G$ has an inverse, f^{-1}, such that $ff^{-1} = e$.*

A group is **finite** if it has a finite number of elements. Otherwise it is infinite.

Definition C.1.2. *The **order** of a group G is the number of elements of G.*

[1] This rule is sometimes called multiplication law of the group.

Definition C.1.3. *A group H whose elements are all elements of a group G is called a* ***subgroup*** *of G.*

Definition C.1.4. ***An Abelian group*** *is one in which the multiplication law is commutative*

$$g_1 g_2 = g_2 g_1, \qquad \forall g_1, g_2 \in G.$$

Definition C.1.5. ***A Representation*** *of G is a mapping, D, of the elements of G onto a set of linear operators with the following properties:*

- *$D(e) = 1$, where 1 is the identity operator in the space on which the linear operators act.*
- *The group multiplication law is mapped onto the natural multiplication in the linear space on which the linear operators act, i.e. $D(g_1)D(g_2) = D(g_1 g_2)$.*

The **dimension** of the representation is the dimension of the space on which it acts.

Definition C.1.6. *A representation is* ***reducible*** *if it has an invariant subspace, which means that the action of any $D(g)$ on any vector in the subspace is still in the subspace. In terms of a projection operator P onto the subspace this condition can be written as*

$$PD(g)P = D(g)P, \qquad \forall g \in G.$$

A representation is **irreducible** if it is not reducible.

C.2 Lie groups

A continuous group is a group G whose elements $g(\alpha)$ depend smoothly on a set of continuous parameters $\alpha = \{\alpha_a\}_{a=1,\ldots,N}$. If the continuous group is in addition a differentiable manifold, it is called **a Lie group**.

Definition C.2.1. *Given a representation $D(\alpha)$ of a Lie group that depends on a set of N real parameters, we define their* ***generators*** *X_a as*

$$X_a \equiv -\frac{\partial}{\partial \alpha_a} D(\alpha)\bigg|_{\alpha=0}. \tag{C.2.1}$$

We normally refer to the **dimension of a Lie group**, dim G, as the number N of generators. This should not be confused with the dimension of a certain representation of the group, labeled by R, which we call dim R.

Group generators are very useful because they keep all the information of the group but, unlike the group elements, they form a vector space, as they can be added together and multiplied by real numbers. They also satisfy some important relations, as it is shown in the following theorem.

Theorem 1. *Generators of a group form a closed commutator algebra, which means*

$$[X_a, X_b] = f_{abc} X_c, \tag{C.2.2}$$

where f_{abc} are constants called the structure constants of the group.

Structure constants are the same for all different representations, as they simply encode the multiplication law of the group. The fact that, in general, generators do not commute, arises from the non-commutativity of the multiplication law. For unitary representations, the structure constants f_{abc} have two important properties

- They are real.
- They are completely antisymmetric.

A proof of these properties can be found in [36]. Note that for an Abelian Lie group (such as $U(1)$, the group of phase transformations), the commutativity property implies that all the structure constants of the group are zero.

The superposition of generators, $\alpha^a X_a$, which are closed under commutation, is the general element of what we call the Lie algebra.

Definition C.2.2. *A **Lie algebra** is a vector space $\mathfrak{g}$ equipped with an alternating bilinear map*

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g}; \\ (x, y) &\mapsto [x, y], \end{aligned}$$

satisfying the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{C.2.3}$$

We stated before that generators keep all the information of the group, without specifyng in which manner they are related. The following theorem provides this relation.

Theorem 2. *The relation between an element of G in a representation $D(\alpha)$ and its corresponding element $\alpha^a X_a$ of the Lie algebra $\mathfrak{g}$ is given by exponentiation*

$$D(\alpha) = e^{-\alpha^a X_a}. \tag{C.2.4}$$

This means that, for unitary representations, in which $D^{-1} = e^{\alpha^a X_a} = e^{-\alpha^a X_a^\dagger} = D^\dagger$, the generators are anti-Hermitian, $X_a^\dagger = -X_a$. [2]

[2] It is common to find in the literature a different convention in the definition (C.2.1) of group generators, which includes a factor i. With this, $D(\alpha) = e^{i\alpha^a X_a}$ and thus generators are Hermitian for unitary representations, $X_a^\dagger = X_a$, instead of anti-Hermitian.

One important representation is the **adjoint representation**, of dimension dim $R =$ dim G, in which the generators are related to the structure constants by $(X_a)^d{}_e \equiv f_{aed}$ [3]. Note that d and e denote row and column indices of matrix X_a, respectively. We now prove that this is indeed a representation, i.e. that they satisfy (C.2.2)

Proof. The Jacobi relation (C.2.3) for the generators can be given in terms of the structure constants if one uses (C.2.2):

$$\begin{aligned} 0 &= [X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] \\ &= f_{bcd} f_{ade} X_e + f_{cad} f_{bde} X_e + f_{abd} f_{cde} X_e \to f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} = 0. \end{aligned} \tag{C.2.5}$$

We can now express (C.2.5) in terms of matrices X_a of the adjoint representation, taking into account the property $f_{cad} = -f_{acd}$

$$(X_a)^e{}_d (X_b)^d{}_c - (X_b)^e{}_d (X_a)^d{}_c = -f_{cde} f_{abd} = f_{abd} (X_d)^e{}_c. \tag{C.2.6}$$

This is the e row and the c column matrix element of

$$X_a X_b - X_b X_a = [X_a, X_b] = f_{abd} X_d. \tag{C.2.7}$$

□

In this representation, we can always pick a basis of the Lie algebra in which the generators are trace orthogonal:

$$\mathrm{Tr}(X_a X_b) = -c\delta_{ab}, \tag{C.2.8}$$

for a positive constant c.

C.2.1 Special Unitary group $SU(N)$

$SU(N)$ groups are very important for the Standard Model of particle physics. In particular, $SU(2)$ and $SU(3)$ are the symmetry groups of the ElectroWeak theory and Quantum Chromodynamics (QCD), respectively.

$SU(N)$ is the special case of a more general group called the Unitary group $U(N)$ (we follow [37]). This is the group of $N \times N$ complex matrices A that are unitary

$$AA^\dagger = \mathbb{1}, \tag{C.2.9}$$

where $A^\dagger = (A^T)^*$ is the Hermitean conjugate of A. Because of (C.2.9) we see that

$$\det(AA^\dagger) = (\det A)(\det A^T)^* = (\det A)(\det A)^* = |\det A|^2 = 1. \tag{C.2.10}$$

[3] In the literature it is also typical to find an extra factor $-i$ in the definition of adjoint representation.

This means that det$A$ has unit modulus, and we can further impose that det$A = 1$. The group of $N \times N$ complex unitary matrices with the restriction det$A = 1$ is what we call $SU(N)$, which is a subgroup of $U(N)$. The generators of this group are $N^2 - 1$ traceless matrices. The fact that the dimension of the group is dim $SU(N) = N^2 - 1$ can be proven as follows:

Proof. We need to find the number of independent generators (i.e., the dimension of the group). We can count how many independent real matrix entries the generators have. We first proceed in this manner for $U(1)$.

As we have previously discussed, the unitarity of the group means that the generators are anti-Hermitian, $X^\dagger = -X$. The i^{th} row and the j^{th} column of this matrix condition is expressed as $X^*_{ji} = X_{ij}$. Thus, for the entries on the diagonal

$$X_{ii} = -X^*_{ii}, \tag{C.2.11}$$

meaning that the diagonal entry is purely imaginary. On the other hand, the entries above the diagonal are the complex conjugates of the corresponding entries below the diagonal. Taking into account those two restrictions, the number of independent real entries is

$$\dim\, U(N) = \frac{2N}{2} + \frac{2N(N-1)}{2} = N^2. \tag{C.2.12}$$

For the case of $SU(N)$ one needs to impose the extra real condition $\det A = 1$, which means

$$\dim\, SU(N) = N^2 - 1. \tag{C.2.13}$$

□

SU(2)

The $SU(2)$ group has dimension dim $SU(2) = 2^2 - 1 = 3$. Therefore, it possesses 3 parameters and 3 generators. The generators are given by $\{X_k = i\sigma_k\}_{k=1,2,3}$, where σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.2.14}$$

Remembering the useful relation that the Pauli matrices satisfy

$$\sigma_i \sigma_j = i\varepsilon_{ijk} + \delta_{ij}\mathbb{1}, \tag{C.2.15}$$

it is immediate to show that $[X_i, X_j] = -2\varepsilon_{ijk}X_k$, so that the structure constants of $SU(2)$ are

$$f_{ijk} = -2\varepsilon_{ijk}. \tag{C.2.16}$$

Although we are not going to discuss $SU(3)$ group, it is worth saying that the 8 generators of these groups are proportional to the *Gell-Mann* matrices, which serve to study the internal rotations of the gluon fields.

C.2.2 Orthogonal group $O(N)$

$O(N)$ is the group of $N \times N$ real matrices R that are orthogonal

$$RR^T = \mathbb{1}. \tag{C.2.17}$$

Orthogonal matrices represent isometries, that is to say, transformations that preserve the distances [4] Examples of isometries are rotations, reflexions, etc. The determinant of those matrices satisfies $\det R = \pm 1$, since

$$\det(RR^T) = (\det R)(\det R^T) = (\det R)^2 = 1. \tag{C.2.18}$$

We can restrict to the subset of matrices for which $\det A = +1$. They represent proper rotations. The Special Orthogonal group, $SO(N)$, is the group of $N \times N$ real orthogonal matrices R with unit determinant, and it is a subgroup of $O(N)$. The dimension of this group is dim $SO(N) = N(N-1)/2$.

Proof. The dimension of $SO(N)$ can be computed by counting the number of independent equations that (C.2.17) imposes on a general real $N \times N$ matrix (note that $\det R = +1$ does not impose any extra real condition). As RR^T is symmetric, then (C.2.17) contains $N(N+1)/2$ independent equations. This means that

$$\dim SO(N) = N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}. \tag{C.2.19}$$

□

Now, in order to introduce the Lie algebra of the $SO(N)$ group, we expand the i^{th} row and the j^{th} column of the matrix R up to first order in parameter ε:

$$R^i{}_j = \left(e^{-\varepsilon r}\right)^i{}_j = \delta^i_j - \varepsilon r^i{}_j + O(\varepsilon^2), \tag{C.2.20}$$

where $r^i{}_j$ is the matrix generator. In order for (C.2.17) to be fulfilled, or equivalently, $R^i{}_k R^i{}_l = \delta_{kl}$, the generator needs to be antisymmetric $r^i_j = -r^j_i$:

$$\begin{aligned} \left[\delta^i_k - \varepsilon r^i{}_k\right]\left[\delta^i_l - \varepsilon r^i{}_l\right] &= \delta^i_k \delta^i_l - \varepsilon\left[r^i{}_l \delta^i_k + \delta^i_l r^i{}_k\right] + O(\varepsilon^2) \\ &\simeq \delta_{lk} - \varepsilon\left[r^k{}_l + r^l{}_k\right] = \delta_{lk}. \end{aligned}$$

The basis for the Lie algebra of $SO(N)$ is formed by the $N(N-1)/2$ generators r. A useful basis for the Lie algebra is given by

$$r_{[\hat{i}\hat{j}]}{}^i{}_j = \delta^i_{\hat{i}} \delta_{\hat{j}j} - \delta^i_{\hat{j}} \delta_{\hat{i}j} = -r_{[\hat{j}\hat{i}]}{}^i{}_j. \tag{C.2.21}$$

[4] This can be seen from the fact that the scalar product is preserved. Taking (C.2.17) into account, note that a transformed vector $u' = Au$ has the same norm as u, since $|u'|^2 = u^T R^T R u = |u|^2$.

Indices $\hat{\imath}$, $\hat{\jmath}$ label the generators, whereas indices i, j label the matrix elements. Both pairs of indices run over $N(N-1)/2$ independent values [5].

The conmutators of these defined generators are:

$$\begin{aligned}
[r_{[\hat{\imath}\hat{\jmath}]}, r_{[\hat{k}\hat{l}]}] &= [\delta^i_{\hat{\imath}}\delta_{\hat{\jmath}j} - \delta^i_{\hat{\jmath}}\delta_{\hat{\imath}j}][\delta^j_{\hat{k}}\delta_{\hat{l}k} - \delta^j_{\hat{l}}\delta_{\hat{k}k}] - [\delta^i_{\hat{k}}\delta_{\hat{l}j} - \delta^i_{\hat{l}}\delta_{\hat{k}j}][\delta^j_{\hat{\imath}}\delta_{\hat{\jmath}k} - \delta^j_{\hat{\jmath}}\delta_{\hat{\imath}k}] \\
&= \delta^i_{\hat{\imath}} \underbrace{\delta_{\hat{\jmath}j}\delta^j_{\hat{k}}}_{\delta_{\hat{\jmath}\hat{k}}} \delta_{\hat{l}k} - \delta^i_{\hat{\imath}} \underbrace{\delta_{\hat{\jmath}j}\delta^j_{\hat{l}}}_{\delta_{\hat{\jmath}\hat{l}}} \delta_{\hat{k}k} - \delta^i_{\hat{\jmath}} \underbrace{\delta_{\hat{\imath}j}\delta^j_{\hat{k}}}_{\delta_{\hat{\imath}\hat{k}}} \delta_{\hat{l}k} + \delta^i_{\hat{\jmath}} \underbrace{\delta_{\hat{\imath}j}\delta^j_{\hat{l}}}_{\delta_{\hat{\imath}\hat{l}}} \delta_{\hat{k}k} \\
&- \delta^i_{\hat{k}} \underbrace{\delta_{\hat{l}j}\delta^j_{\hat{\imath}}}_{\delta_{\hat{l}\hat{\imath}}} \delta_{\hat{\jmath}k} + \delta^i_{\hat{k}} \underbrace{\delta_{\hat{l}j}\delta^j_{\hat{\jmath}}}_{\delta_{\hat{l}\hat{\jmath}}} \delta_{\hat{\imath}k} + \delta^i_{\hat{l}} \underbrace{\delta_{\hat{k}j}\delta^j_{\hat{\imath}}}_{\delta_{\hat{k}\hat{\imath}}} \delta_{\hat{\jmath}k} - \delta^i_{\hat{l}} \underbrace{\delta_{\hat{k}j}\delta^j_{\hat{\jmath}}}_{\delta_{\hat{k}\hat{\jmath}}} \delta_{\hat{\imath}k} \\
&= \boxed{\delta_{\hat{\jmath}\hat{k}} r_{[\hat{\imath}\hat{l}]} - \delta_{\hat{\imath}\hat{k}} r_{[\hat{\jmath}\hat{l}]} - \delta_{\hat{\jmath}\hat{l}} r_{[\hat{\imath}\hat{k}]} - \delta_{\hat{\imath}\hat{l}} r_{[\hat{\jmath}\hat{k}]}}\,. \qquad \text{(C.2.22)}
\end{aligned}$$

This equation specifies the structure constants of the Lie algebra.

SO(3)

We are going to check that for $N = 3$, expression (C.2.21) reduces to the generators of the well-known rotation matrices in 3-dimensional space. For this case we have $3(3-1)/2 = 3$ generators, that we label with the 3 independent antisymmetric combinations $\left\{[\hat{\imath}\hat{\jmath}] = 21, 32, 13\right\}$. Using (C.2.21) we obtain the explicit form of the generators

$$r_{21} \equiv X_A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r_{32} \equiv X_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad r_{13} \equiv X_C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \qquad \text{(C.2.23)}$$

We are going to compute $e^{-\theta X_A}$ for one parameter θ among the three, to find that this leads to a rotation around the axis corresponding to the 3-direction of space [6].

$$\begin{aligned}
R_A =& e^{-\theta X_A} = \mathbb{1} - \theta X_A + \frac{\theta^2 X_A^2}{2} - \frac{\theta^3 X_A^3}{6} + ... = \begin{pmatrix} 1-\theta^2/2+... & -\theta+\theta^3/6+... & 0 \\ \theta-\theta^3/6+... & 1-\theta^2/2+... & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
=& \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \text{(C.2.24)}
\end{aligned}$$

This is indeed the matrix of a rotation about the 3-direction (or z direction) of space.

[5] This is the number of independent component of an antisymmetric $N \times N$ matrix.

[6] Here we have used the definition of the exponential of a matrix, $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$, being $A^0 \equiv \mathbb{1}$ the identity.

O(p, q)

The defining condition (C.2.17) for the $O(N)$ group can also be written as

$$R^i{}_k \delta_{ij} R^j{}_l = \delta_{kl}. \tag{C.2.25}$$

This equation expresses the fact that the transformations of the $O(N)$ group leaves the Euclidean metric $g_{ij} = \delta_{ij}$ invariant. But we can think of a more general group that leaves the following diagonal metric

$$g_{ij} = \text{diag}(\underbrace{-, ..., -}_{q\ \text{times}}, \underbrace{+, ..., +}_{p\ \text{times}}) \tag{C.2.26}$$

invariant. We normally say that a metric of this kind has (p, q) signature. The pseudo-orthogonal group, $O(p, q)$, is the group of $N \times N$ real matrices, with $N = p + q$, that leaves a metric g_{ij} of signature (p, q) invariant

$$R^i{}_k g_{ij} R^j{}_l = g_{kl}. \tag{C.2.27}$$

The dimension of this group is dim $O(p, q) = N(N - 1)/2$. This group is specially important for the study of the Lorentz group, as we will see soon.

C.2.3 The Lorentz and Poincaré groups

The Lorentz group deals with the space-time symmetry of all known fundamental laws of Nature. Here we review the most important concepts.

The **Lorentz group** is the set of homogeneous linear transformations of coordinates in D-dimensional Minkowski spacetime that preserve the norm of any vector [30]. We write the transformations as

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu, \qquad x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu. \tag{C.2.28}$$

By requiring that $|x|^2 = x^\mu \eta_{\mu\nu} x^\nu = x'^\mu \eta_{\mu\nu} x'^\nu = |x'|^2$, we arrive at the condition $x'^\sigma \Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho x'^\rho = x'^\mu \eta_{\mu\nu} x'^\nu$, which in turn implies:

$$\boxed{\Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}}. \tag{C.2.29}$$

This is the property that characterizes Λ matrices. Note that (C.2.29) is the defining condition of the pseudo-orthogonal group $O(D - 1, 1)$. We proceed to see more consequences of this equation. In matrix form, (C.2.29) is $\eta = \Lambda^T \eta \Lambda$. Taking determinants and using $\det\Lambda^T = \det\Lambda$ we have:

$$\det\eta = \det\Lambda \ \det\eta \ \det\Lambda \quad \to \quad \det\Lambda = \pm 1. \tag{C.2.30}$$

On the other hand, taking the 00 entry of (C.2.29) we realize that:

$$-1 = \eta_{00} = \Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0 = \left(\Lambda^i{}_0\right)^2 - \left(\Lambda^0{}_0\right)^2 \ \to \ \left|\Lambda^0{}_0\right| \geq 1. \tag{C.2.31}$$

	Proper $\det\Lambda = 1$	**Improper** $\det\Lambda = -1$
Ortochronous $\Lambda^0{}_0 \geq 1$	Proper rotations Boosts	Spatial inversion P
Non-ortochronous $\Lambda^0{}_0 \leq -1$	Time inversion T	Spatial and time inversion PT

Table C.1: *Examples of transformations for each of the four categories of the Lorentz group.*

We can classify the transformations of the Lorentz group according to conditions (C.2.30) and (C.2.31) [38]. Lorentz transformations with $\det\Lambda = 1$ (-1) are said to be proper (improper). The case $\Lambda^0{}_0 \geq 1$ (≤ -1) corresponds to orthochronous (non-orthochronous) Lorentz transformations. Examples of these transformations are shown in Table C.2.3

The property $\Lambda^0{}_0 \geq 1$ excludes the possibility of time inversions $T = \mathrm{diag}(-1, 1, 1, 1)$, whereas the property $\det\Lambda = 1$ excludes the possibility of spatial inversions $P = \mathrm{diag}(1, -1, -1, -1)$. Lorentz transformations that satisfy either $\det\Lambda = -1$ or $\Lambda^0{}_0 \leq -1$ or both simultaneously are said to be the disconnected components of the Lorentz group.

The set of Lorentz transformations characterized by the restrictions $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$, (i.e. transformations that preserve orientation and direction of time) is denoted by $SO^+(D-1, 1)$ and it is a subgroup of $O(D-1, 1)$. This is the connected part of the Lorentz group. [7] We are specially interested in $SO^+(D-1, 1)$, not only because it includes proper rotations and boosts (these are, changes to inertial reference frames), but also because it is possible to obtain its Lie algebra. For this reason, unless it is specified, when we speak about the Lorentz group, we will always refer to $SO^+(D-1, 1)$.

There are some useful relations that can be deduced from (C.2.28) and (C.2.29):

$$\Lambda_{\mu\nu} = (\Lambda^{-1})_{\nu\mu}, \qquad \Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu}, \tag{C.2.32}$$

$$x'_{\mu} = (\Lambda^{-1})_{\mu}{}^{\nu} x_{\nu} = x_{\nu} \Lambda^{\nu}{}_{\mu}. \tag{C.2.33}$$

Proof. We take equation (C.2.29) and lower the index of the left Λ matrix using the metric, yielding $\Lambda_{\nu\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$. Then we raise the ρ index at each side and we get:

$$\Lambda_{\nu}{}^{\rho}\Lambda^{\nu}{}_{\sigma} = \eta^{\rho}{}_{\sigma} = \eta^{\rho\nu}\eta_{\nu\sigma} = \delta^{\rho}_{\sigma}. \tag{C.2.34}$$

By comparing this expression with $(\Lambda^{-1})^{\rho}{}_{\nu}\Lambda^{\nu}{}_{\sigma} = \delta^{\rho}_{\sigma}$ we infer $(\Lambda^{-1})^{\rho}{}_{\nu} = \Lambda_{\nu}{}^{\rho}$ or, equivalently $(\Lambda^{-1})_{\nu}{}^{\rho} = \Lambda^{\rho}{}_{\nu}$. Now we take (C.2.29) and lower the index of the right Λ

[7] The connected part of a group is the one that can be given as the exponentiation of an algebra. If this is not possible, we talk about disconnected components of the group.

matrix with $\eta_{\mu\nu}$, to obtain $\Lambda^{\mu}{}_{\rho}\Lambda_{\mu\sigma} = \eta_{\rho\sigma}$. After raising the ρ index at each side (by contracting with the metric), we arrive at:

$$\Lambda^{\mu\rho}\Lambda_{\mu\sigma} = \eta^{\rho}{}_{\sigma} = \delta^{\rho}{}_{\sigma}. \tag{C.2.35}$$

Then, by comparing this expression with $\Lambda^{\mu\rho}(\Lambda^{-1})_{\sigma\mu} = \delta^{\rho}{}_{\sigma}$ we infer that $(\Lambda^{-1})_{\sigma\mu} = \Lambda_{\mu\sigma}$. Finally, we proceed as:

$$x'^{\mu} = \eta_{\mu\nu}x'^{\nu} = \eta_{\mu\nu}(\Lambda^{-1})^{\nu}{}_{\rho}x^{\rho} = \eta_{\mu\nu}(\Lambda^{-1})^{\nu}{}_{\rho}\eta^{\rho\tau}x_{\tau} = (\Lambda^{-1})_{\mu}{}^{\tau}x_{\tau} \underbrace{=}_{\tau\to\nu} (\Lambda^{-1})_{\mu}{}^{\nu}x_{\nu}.$$

□

So as to introduce **the Lie algebra of the Lorentz group**, we expand the transformation Λ up to first order in a, yet unspecified, parameter λ:

$$\Lambda^{\mu}{}_{\nu} = (e^{\lambda m})^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \lambda m^{\mu}{}_{\nu} + O(\lambda^2). \tag{C.2.36}$$

Following the same reasoning we did for the generators of the $SO(N)$ group, (C.2.36) satisfies (C.2.29) to first order in λ as long as the generator is antisymmetric in its two lower indices $m_{\mu\nu} = -m_{\nu\mu}$. The basis of the Lie algebra is thus formed by the $D(D-1)/2$ independent generators $m_{\mu\nu}$. A useful representation is

$$m_{[\rho\sigma]}{}^{\mu}{}_{\nu} \equiv \delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\rho\nu} = -m_{[\sigma\rho]}{}^{\mu}{}_{\nu}. \tag{C.2.37}$$

Again, the indices in brackets $[\rho\sigma]$ label the generators whereas the indices μ and ν label the matrix elements. Both pairs of indices run over $D(D-1)/2$ values. We need to label $D(D-1)/2$ real parameters λ of the algebra, and a way of doing so is using an antisymmetric pair of indices so that $\lambda^{\rho\sigma} = -\lambda^{\sigma\rho}$. With all of this in mind, an ortochronous proper Lorentz transformation can be written as

$$\Lambda = e^{\frac{1}{2}\lambda^{\rho\sigma}m_{[\rho\sigma]}}, \tag{C.2.38}$$

so that

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \frac{\lambda^{\rho\sigma}}{2}(\delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\rho\nu}) + O(\lambda^2) = \delta^{\mu}_{\nu} + \lambda^{\mu}{}_{\nu} + O(\lambda^2). \tag{C.2.39}$$

In an analogous way as for the generators of the $SO(N)$ group, the commutator of the generators in (C.2.37) is shown to be

$$[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \eta_{\nu\rho}m_{[\mu\sigma]} - \eta_{\mu\rho}m_{[\nu\sigma]} - \eta_{\nu\sigma}m_{[\mu\rho]} + \eta_{\mu\sigma}m_{[\nu\rho]}. \tag{C.2.40}$$

Equations (C.2.40) specify the structure constants of the Lie algebra, which we can read off as:

$$f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} = 8\eta_{[\rho[\nu}\delta^{[\kappa}_{\mu]}\delta^{\tau]}_{\sigma]}. \tag{C.2.41}$$

Proof. If we expand the right hand side of expression (C.2.41) for each antisymmetric pair of indices, we will get eight terms for the structure constants. Introducing the structure constants in their defining expression, we obtain

$$\begin{aligned}
&[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \frac{1}{2} f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} m_{[\kappa\tau]} = \frac{m_{[\kappa\tau]}}{2} \times \\
&\times \left(\eta_{\rho\nu}\delta^\kappa_\mu\delta^\tau_\sigma - \eta_{\sigma\nu}\delta^\kappa_\mu\delta^\tau_\rho - \eta_{\rho\mu}\delta^\kappa_\nu\delta^\tau_\sigma + \eta_{\sigma\mu}\delta^\kappa_\nu\delta^\tau_\rho - \eta_{\rho\nu}\delta^\tau_\mu\delta^\kappa_\sigma + \eta_{\sigma\nu}\delta^\tau_\mu\delta^\kappa_\rho + \eta_{\rho\mu}\delta^\tau_\nu\delta^\kappa_\sigma - \eta_{\sigma\mu}\delta^\tau_\nu\delta^\kappa_\rho \right) \\
&= \frac{\eta_{\rho\nu}}{2} \underbrace{(m_{[\mu\sigma]} - m_{[\sigma\mu]})}_{2m_{[\mu\sigma]}} - \frac{\eta_{\sigma\nu}}{2} \underbrace{(m_{[\mu\rho]} - m_{[\rho\mu]})}_{2m_{[\mu\rho]}} - \frac{\eta_{\rho\mu}}{2} \underbrace{(m_{[\nu\sigma]} - m_{[\sigma\nu]})}_{2m_{[\nu\sigma]}} + \frac{\eta_{\sigma\mu}}{2} \underbrace{(m_{[\nu\rho]} - m_{[\rho\nu]})}_{2m_{[\nu\rho]}} \\
&= \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]}. \qquad \text{(C.2.42)}
\end{aligned}$$

□

Let us study now how the Lorentz transformations are implemented when acting on fields. The following differential operator is important for studying the infinitesimal Lorentz variations on fields.

$$L_{[\rho\sigma]} \equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho = -L_{[\sigma\rho]}, \qquad \text{(C.2.43)}$$

These operators have a commutator algebra that it is isomorphic to that of $m_{[\rho\sigma]}$.

Proof. We will make use of the fact that $\partial_\nu x_\mu = \partial_\nu(\eta_{\mu\alpha} x^\alpha) = \eta_{\mu\alpha}\delta^\alpha_\nu = \eta_{\mu\nu}$. We compute the following:

$$\begin{aligned}
&[L_{[\mu\nu]}, L_{[\rho\sigma]}] = [x_\mu\partial_\nu - x_\nu\partial_\mu, x_\rho\partial_\sigma - x_\sigma\partial_\rho] = x_\mu\partial_\nu(x_\rho\partial_\sigma) - x_\mu\partial_\nu(x_\sigma\partial_\rho) - x_\nu\partial_\mu(x_\rho\partial_\sigma) \\
&+ x_\nu\partial_\mu(x_\sigma\partial_\rho) - x_\rho\partial_\sigma(x_\mu\partial_\nu) + x_\rho\partial_\sigma(x_\nu\partial_\mu) + x_\sigma\partial_\rho(x_\mu\partial_\nu) - x_\sigma\partial_\rho(x_\nu\partial_\mu). \qquad \text{(C.2.44)}
\end{aligned}$$

We note that each of those terms will give another two terms when taking the derivative: one will contain the metric and the other a cross-derivative. For example, the first term will give $x_\mu\eta_{\nu\rho}\partial_\sigma + x_\mu x_\rho\partial_\nu\partial_\sigma$. We notice that all the resulting terms with cross-derivatives cancel each other, so we finally get:

$$\begin{aligned}
&[L_{[\mu\nu]}, L_{[\rho\sigma]}] = x_\mu\eta_{\nu\rho}\partial_\sigma - x_\mu\eta_{\nu\sigma}\partial_\rho - x_\nu\eta_{\mu\rho}\partial_\sigma + x_\nu\eta_{\mu\sigma}\partial_\rho - x_\rho\eta_{\sigma\mu}\partial_\nu + \\
&+ x_\rho\eta_{\sigma\nu}\partial_\mu + x_\sigma\eta_{\rho\mu}\partial_\nu - x_\sigma\eta_{\rho\nu}\partial_\mu = \eta_{\nu\rho}L_{[\mu\sigma]} - \eta_{\mu\rho}L_{[\nu\sigma]} - \eta_{\nu\sigma}L_{[\mu\rho]} + \eta_{\mu\sigma}L_{[\nu\rho]}.
\end{aligned}$$

□

We start considering the **transformations on scalar fields** $\phi(x)$, which are the simplest kinds of fields. A scalar field $\phi(x)$ is transformed under the mapping $U(\lambda)$ as

$$\phi(x) \to \phi'(x) = U(\Lambda)\phi(x) = \phi(\Lambda x). \qquad \text{(C.2.45)}$$

Λx is a short-hand writing of $\Lambda_\nu{}^\mu x^\nu$. We need an explicit form for the mapping $U(\Lambda)$. The differential operator

$$U(\Lambda) = e^{-\frac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}} \qquad \text{(C.2.46)}$$

satisfies (C.2.45). This is not surprising, as we just showed that $L_{[\rho\sigma]}$ was an implementation of the Lie algebra in the form of differential operators, acting on scalar fuctions rather than on coordinates. By expanding the exponential and neglecting terms above first order we get the infinitesimal variation

$$\delta\phi(x) = U(\Lambda)\phi(x) - \phi(x) = -\frac{1}{2}\lambda^{\mu\nu}L_{\mu\nu}\phi(x) + O(\lambda^2). \tag{C.2.47}$$

Proof. We are going to prove that (C.2.46) is the correct mapping for scalar fields up to first order, i.e. it satisfies (C.2.47), so we neglect $O(\lambda^2)$ terms during this calculation. First note that

$$U(\Lambda)\phi(x) = \phi(\Lambda^{\mu}{}_{\nu}x^{\nu}) = \phi\left([\delta^{\mu}_{\nu} + \lambda^{\mu}{}_{\nu}]x^{\nu}\right) = \phi(x^{\mu} + \lambda^{\mu\nu}x_{\nu}) \tag{C.2.48}$$

Now we are going to show that $\phi(x^{\mu} + \lambda^{\mu\nu}x_{\nu}) = \phi(x^{\mu}) - \frac{1}{2}\lambda^{\rho\sigma}L_{\rho\sigma}\phi(x^{\mu})$. First, we notice that by performing a Taylor expansion up to first order $\phi(x^{\mu} + \lambda^{\mu\nu}x_{\nu}) = \phi(x^{\mu}) + \lambda^{\sigma\rho}x_{\rho}\partial_{\sigma}\phi(x^{\mu}) + O(\lambda^2)$. Now this last term can be expressed in the following way:

$$\lambda^{\sigma\rho}x_{\rho}\partial_{\sigma}\phi(x^{\mu}) = \frac{1}{2}\left(\lambda^{\sigma\rho}x_{\rho}\partial_{\sigma}\phi(x^{\mu}) + \lambda^{\sigma\rho}x_{\rho}\partial_{\sigma}\phi(x^{\mu})\right). \tag{C.2.49}$$

Using the antisymmetry of $\lambda^{\sigma\rho}$ for the second term and relabeling the dummy indices we notice that:

$$\lambda^{\sigma\rho}x_{\rho}\partial_{\sigma}\phi(x^{\mu}) = \frac{\lambda^{\sigma\rho}}{2}(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho})\phi(x^{\mu}) = -\frac{1}{2}\lambda^{\rho\sigma}L_{\rho\sigma}\phi(x^{\mu}). \tag{C.2.50}$$

□

The next case we consider is **the transformation rules for covariant and contravariant vector fields**. The transformation for a general covariant field $W_{\mu}(x)$ is:

$$W_{\mu}(x) \;\rightarrow\; W'_{\mu}(x) = U(\Lambda)W_{\mu}(x) = (\Lambda^{-1})_{\mu}{}^{\nu}W_{\nu}(\Lambda x), \tag{C.2.51}$$

whereas for a general contravariant vector field $V^{\mu}(x)$ the transformation is of the form:

$$V^{\mu}(x) \;\rightarrow V'^{\mu}(x) = U(\Lambda)V^{\mu}(x) = \;(\Lambda^{-1})^{\mu}{}_{\sigma}V^{\sigma}(\Lambda x). \tag{C.2.52}$$

These definitions are consistent. For example, for the scalar quantity $V^{\mu}(x)W_{\mu}(x)$, the equation $V^{\mu}(x)W_{\mu}(x) = V^{\mu}(\Lambda x)W_{\mu}(\Lambda x)$ is satisfied, as it was required. A difference with respect to the case of scalar fields, is that the transformation now affects not only in the spacetime coordinates x^{μ}, but also the fields themselves. We need now to determine the correct form of the mapping $U(\Lambda)$, which is a matrix. The appropiate form is given by (omitting matrix indices for simplicity)

$$U(\Lambda) = e^{-\frac{1}{2}\lambda^{\rho\sigma}J_{[\rho\sigma]}}, \tag{C.2.53}$$

where the $J_{[\rho\sigma]}$ are defined to act on contravariant and covariant vector fields as

$$J_{[\rho\sigma]}V^{\mu}(x) = (L_{[\rho\sigma]}\delta^{\mu}_{\ \nu} + m_{[\rho\sigma]}{}^{\mu}{}_{\nu})V^{\nu}(x), \tag{C.2.54}$$
$$J_{[\rho\sigma]}W_{\nu}(x) = (L_{[\rho\sigma]}\delta_{\nu}^{\ \mu} + m_{[\rho\sigma]\nu}{}^{\mu})W_{\mu}(x). \tag{C.2.55}$$

Again, by expanding the exponential and neglecting second order terms, we get the infinitesimal variation $\delta V^{\mu} = U(\Lambda)V^{\mu} - V^{\mu} = -\frac{1}{2}\lambda^{\rho\sigma}J_{[\rho\sigma]}V^{\mu} + O(\lambda^2)$. Note that we are always omitting the matrix indices and writing them only just when we are interested in showing the explicit form of the transformation.

The **Poincaré Group** is defined by adding global spacetime translations to the Lorentz Group. In this case, the non-homogeneous transformations of coordinates are given by :

$$x'^{\mu} = (\Lambda^{-1})^{\mu}_{\ \nu}(x^{\nu} - a^{\nu}). \tag{C.2.56}$$

For spacetime translations $x^{\mu} \to x'^{\mu} = x^{\mu} - a^{\mu}$, the mapping is more easily implemented. We consider its action on $\{\psi^i(x)\}_{i=1,\ldots,n}$, which are a set of scalar fields or the different parts of a multi-component field

$$\psi^i(x) \ \to \ \psi'^i(x) = \psi^i(x+a) = U(a)\psi^i(x), \tag{C.2.57}$$
$$\text{being} \ \ U(a) = e^{a^{\mu}P_{\mu}} \quad \text{with} \ \ P_{\mu} = \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}. \tag{C.2.58}$$

Here P_{μ} is the generator and $U(a)$ is called the translation operator. The mappings of Poincaré group are given by operators $U(a,\Lambda) \equiv U(\Lambda)U(a)$, which act as follows (we are omitting matrix multiplication indices):

$$\psi(x) \ \to \ \psi'(x) \equiv U(a,\Lambda)\psi(x) = U(\Lambda)U(a)\psi(x) = e^{-\frac{1}{2}\lambda^{\rho\sigma}m_{[\rho\sigma]}}\psi(\Lambda x + a). \tag{C.2.59}$$

There are $D(D-1)/2$ generators $J_{[\rho\sigma]}$ and D generators P_{μ}, so in total the Poincaré group has $D(D+1)/2$ generators. The infinitesimal variation of the fields ψ^i is given by

$$\delta\psi = U(\Lambda)U(a)\psi(x) - \psi(x) = \left[a^{\mu}P_{\mu} - \frac{1}{2}\lambda^{\rho\sigma}J_{[\rho\sigma]}\right]\psi(x) + \text{higher order terms} \tag{C.2.60}$$

The Lie algebra is specified by the following commutation relations

$$[J_{[\mu\nu]}, J_{[\rho\sigma]}] = \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{[\nu\sigma]} - \eta_{\nu\sigma}J_{[\mu\rho]} + \eta_{\mu\sigma}J_{[\nu\rho]}, \tag{C.2.61}$$
$$\left[J_{[\rho\sigma]}, P_{\mu}\right] = P_{\rho}\eta_{\sigma\mu} - P_{\sigma}\eta_{\rho\mu}, \tag{C.2.62}$$
$$[P_{\mu}, P_{\nu}] = 0. \tag{C.2.63}$$

Proof. The last commutator is the simplest: its zero value comes from the fact that the partial derivatives are always assumed to commute. The commutator (C.2.61) is isomorphic to that of $m_{[\rho\sigma]}$, and that is why we say this is another representation of the generators of the Lorentz group. This commutation relation is rapidly seen to be satisfied, as $J_{[\mu\nu]}$ is formed by $m_{[\rho\sigma]}$ and $L_{[\rho\sigma]}$, which obey the same commutation algebra (as we already showed). We proceed to derive (C.2.62)

$$\begin{aligned}\left[J_{[\rho\sigma]}, P_\mu\right] &= J_{[\rho\sigma]}P_\mu - P_\mu J_{[\rho\sigma]} = J_{[\rho\sigma]}P_\mu - P_\nu L_{[\rho\sigma]} \\ &= \cancel{L_{[\rho\sigma]}P_\nu} + (m_{[\rho\sigma]})_\nu{}^\mu P_\mu - \cancel{P_\nu L_{[\rho\sigma]}} = [\delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\rho\nu}]P_\mu \\ &= P_\rho \eta_{\sigma\mu} - P_\sigma \eta_{\rho\mu}.\end{aligned}$$

□

C.2.3.1 The Lorentz group for $D = 4$

We are going to check that for the more familiar case of spacetime dimension $D = 4$, expression (C.2.37) reduces to the generators of the well-known boosts and rotations matrices. For this case we have $4(4-1)/2 = 6$ generators, that we label with the 6 independent antisymmetric combinations $\{[\rho\sigma] = 01, 02, 03, 21, 32, 13\}$. Using (C.2.37) we obtain the explicit form of the commutators

$$m_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad m_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{C.2.64}$$

$$m_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{C.2.65}$$

The three generators in (C.2.64) correspond to the proper rotations in the three spatial directions. They are just those we already found at (C.2.23), with an extra time dimension which is unaffected by the rotation. The other three generators at (C.2.65) correspond to boosts in the three spatial directions. In Appendix E we have proved shown with the help of Mathematica that these generators actually satisfy (C.2.40). If we call $\{J_i\}_{i=1,2,3}$ to the three rotations and $\{K_i\}_{i=1,2,3}$ to the three boosts, a compact way of writting these antisymmetric combinations is $J_i = -\frac{1}{2}\varepsilon_{ijk}m_{jk}$ and $K_i = m_{[0i]}$. In order to check that (C.2.65) does correspond to boosts, we compute

$e^{-m\rho} \equiv e^{-\rho_{01}m_{01}}$ for one parameter ρ out of the six:

$$\Lambda = e^{-m\rho} = \mathbb{1} - \rho m + \frac{\rho^2 m^2}{2} - \frac{\rho^3 m^3}{6} + ... =$$
$$= \begin{pmatrix} 1+\rho^2/2+... & -\rho-\rho^3/6-... & 0 & 0 \\ -\rho-\rho^3/6-... & 1+\rho^2/2+... & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\rho & -\sinh\rho & 0 & 0 \\ -\sinh\rho & \cosh\rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.2.66}$$

If we now call
$$\cosh\rho = \gamma, \qquad \sinh\rho = \beta\gamma, \tag{C.2.67}$$
where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we see that in (C.2.66) we recover the usual expression for a Lorentz transformation or boost along the 1-direction (or x direction). Notice that the identification (C.2.67) is possible as $\cosh^2\rho - \sinh^2\rho = \gamma^2 - \beta^2\gamma^2 = 1$. The parameter ρ is conventionally called the rapidity.

Here we include the proof that the six generators in (2.1.1), formed by complex linear combinations of the six generators above, satisfy the commutation relations of two independent copies of $\mathfrak{su}(2)$.

Proof. We begin with the proof of the first commutator in (2.1.2). We introduce the defining expression of I_i into its conmutator expression:

$$[I_i, I_j] = \frac{1}{4}\left([J_i, J_j] + [K_j, K_i] + i[K_j, J_i] + i[J_j, K_i]\right). \tag{C.2.68}$$

We compute each commutator separately, using structure constants for the commutator algebra of $m_{[\mu\nu]}$ (see (C.2.40)):

$$\begin{aligned} [J_i, J_j] &= \frac{\epsilon_{irt}\epsilon_{jsu}}{4}[m_{[rt]}, m_{[su]}] = \frac{\epsilon_{irt}\epsilon_{jsu}}{8} f_{[rt][su]}{}^{[ln]} m_{[ln]} = \epsilon_{irt}\epsilon_{jsu}\eta_{[s[t}\delta^{[l}_{r]}\delta^{n]}_{u]} m_{[ln]} \\ &= \frac{1}{2}\epsilon_{jsu}\epsilon_{irt}\eta_{st}m_{ru} = \frac{1}{2}\epsilon_{jsu}\epsilon_{irt}\delta_{st}m_{ru} = \frac{1}{2}\epsilon_{jsu}\epsilon_{irs}m_{[ru]}. \end{aligned} \tag{C.2.69}$$

Since the Levi-Civita symbol only involves spatial indices and no time indices, we can freely raise and lower them to write $\epsilon_{jsu}\epsilon_{irs} = \epsilon_j{}^{su}\epsilon_i{}^r{}_s$. Then, by making use of the identity $\epsilon^{i_1...i_q k_1...k_p}\epsilon_{j_1...j_q k_1...k_p} = -p!q!\ \delta^{i_1...i_q}_{j_1...j_q}$, we realize that:

$$[J_i, J_j] = -\frac{1}{2}\epsilon_j{}^{us}\epsilon_i{}^r{}_s m_{[ru]} = \delta^{ur}_{ji} m_{[ru]} = -\epsilon_{ji}{}^s\epsilon_s{}^{ur} m_{[ru]} = -\frac{1}{2}\epsilon_{ij}{}^s\epsilon_s{}^{ur} m_{[ur]} = \epsilon_{ijs}J_s.$$

Repeating this process for the other commutators, we obtain:

$$\begin{aligned} i[J_j, K_i] &= -\frac{i\epsilon_{jrs}}{2}[m_{[rs]}, m_{[0i]}] = -\frac{i\epsilon_{jrs}}{4} f_{[rs][0i]}{}^{[ln]} m_{[ln]} = -2i\epsilon_{jrs}\eta_{[0s}\delta^{[l}_{r}\delta^{n]}_{i]} m_{[ln]} \\ &= -i\epsilon_{jrs}\left(\eta_{0s}m_{ri} - \eta_{is}m_{r0}\right) = i\epsilon_{jri}m_{r0} = i\epsilon_{jir}m_{[0r]} = i\epsilon_{jir}K_r, \end{aligned} \tag{C.2.70}$$

where in the last step we have set $\epsilon_{jrs}\eta_{0s}$ to zero since s can only take values from 1 to 3. We notice that $i[J_j, K_i] = i[K_j, J_i]$. For the last commutator:

$$[K_j, K_i] = [m_{[0j]}, m_{[0i]}] = \frac{1}{2} f_{[0j][0i]}{}^{[ln]} m_{[ln]} = 4\eta_{[0[j}\delta^{[l}_{0]}\delta^{n]}_{i]} m_{[ln]} = \frac{1}{2}\delta^{[l}_j \delta^{n]}_i m_{[ln]}$$
$$= \frac{1}{2}\epsilon_{jis}\epsilon_{lns} m_{[ln]} = \frac{1}{2}\epsilon_{jis}\epsilon_{sln} m_{[ln]} = -\epsilon_{jis} J_s, \tag{C.2.71}$$

where in the last step we have used $\delta^{[l}_j \delta^{n]}_i = \epsilon^{lns}\epsilon_{jis}$. Finally, inserting everything in (C.2.68), we have:

$$[I_i, I_j] = \frac{1}{4}\left(\epsilon_{ijk} J_k - \epsilon_{jik} J_k + 2i\epsilon_{jik} K_k\right) = \frac{\epsilon_{ijk}}{2}\left(J_k - iK_k\right) = \epsilon_{ijk} I_k. \tag{C.2.72}$$

In the same way, for the third commutator in (2.1.2), we see:

$$[I_i, I'_j] = \frac{1}{4}\left([J_i, J_j] - [K_j, K_i] - i[K_j, J_i] + i[J_j, K_i]\right) = 0. \tag{C.2.73}$$

□

Appendix D

Basic notions of Clifford algebras

The Dirac equation is a first order equation in space and time derivatives that is invariant under Lorentz transformations . Dirac managed this achievement thanks to a set of γ-matrices, satisfying the following property

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}\mathbb{1}. \tag{D.0.1}$$

These γ-matrices generate an specific Clifford algebra, a mathematical structure that had been created by W.K. Clifford half a century before Dirac wrote his famous equation. Clifford algebras are widely used nowdays in geometry, theoretical physics and digital image processing [39].

In the first part of this appendix, we introduce some basic notions of Clifford algebras, whereas in the second part, we study some properties of the spinors as an application of Clifford algebras. These concepts are necessary for the study of Majorana fermions and supersymmetry [30].

D.1 The Clifford algebra

The generating elements

A general and explicit construction of the γ-matrices in arbitrary dimension D can be given in terms of Pauli matrices

$$\begin{aligned}
\gamma^0 &= i\sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ..., \\
\gamma^1 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ..., \\
\gamma^2 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ..., \\
\gamma^3 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes ..., \\
\gamma^4 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \\
... &= ...,
\end{aligned} \tag{D.1.1}$$

where ... means that the construction continues with the same pattern, for increasing dimension D. These γ-matrices all square to $\mathbb{1}$ (except γ^0, which squares to

$-\mathbb{1}$ because of the pressence of the i) and they mutually anti-commute. This follows from the fact that Pauli matrices σ_i square to $\mathbb{1}$ and anti-commute. In Appendix E we have shown with the help of Mathematica that the γ-matrices in (D.1.1) satisfy (D.0.1), as it is required. Let us study now what is the dimension of these matrices for a general spacetime dimension D.

Suppose an even dimension, that is, $D = 2m$ for some natural number m. In order to get $2m$ different γ^μ's, we need to have m Pauli matrices in the tensor products of each γ^μ in (D.1.1). And since the Pauli matrices are 2×2, the representation has dimension $2^m = 2^{D/2}$.

For odd dimension, $D = 2m+1$, one additional matrix γ^{2m+1} is required. However, there is no need to add more Pauli matrices in the tensor products, so we still have m factors in each γ^μ. As there is no increase in the dimensions when going from $D = 2m$ to $D = 2m + 1$, we can conclude that in general the dimension of the representation in (D.1.1) is $2^{[D/2]}$, where $[D/2]$ means the integer part of $D/2$ [1]. Regarding the Hermiticity

- γ^0 is anti-Hermitian, $(\gamma^0)^\dagger = -\gamma^0$.
- γ^i for $i = 1, ..., D-1$, are Hermitian $(\gamma^i)^\dagger = \gamma^i$.

The Hermiticity property can be summarized as

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0, \tag{D.1.2}$$

since $(\gamma^0)^\dagger = (\gamma^0)^2\gamma^0 = -\gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i(\gamma^0)^2 = \gamma^i$. The representations in which (D.1.2) holds are called Hermitian representations. From (D.1.1) we also see that, for a generic dimension D, the γ-matrices are complex.

Up to conjugation, $\gamma'^\mu = S\gamma^\mu S^{-1}$ (where S is any unitary matrix), there is a unique irreducible representation of the Clifford algebra for even dimension. For odd dimension, there are two inequivalent irreducible representations. A proof of this, which uses some well-known results of finite group theory, can be found in [40].

The full Clifford algebra

The complete Clifford algebra is composed of the identity $\mathbb{1}$, the D generating matrices γ^μ, and all independent products of these generating matrices. We need to reject symmetric products, as they reduce to a product containing fewer γ-matrices. This can be seen by looking at (D.0.1), which implies $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbb{1}$ for symmetric

[1] An interesting corollary of this result is that the spacetime dimension D is in general different from the dimension of the γ-matrices. For example, they coincide for $D = 4$, but for $D = 10$, the γ-matrices are 32×32.

products, and therefore $\gamma^\mu\gamma^\nu = 0$ for $\mu \neq \nu$. Thus, only antysimmetric products are considered. We define

$$\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}...\gamma^{\mu_r]} \equiv \frac{1}{r!}\sum_\sigma \gamma^{\mu_1}...\gamma^{\mu_r}, \tag{D.1.3}$$

where σ denotes the set of $r!$ signed permutations of indices $\mu_1, ..., \mu_r$. Because of the antisymmetry, the only non-zero components of those products are

$$\gamma^{\mu_1\mu_2\cdots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_r} \quad \text{where} \quad \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r. \tag{D.1.4}$$

All matrices of the Clifford algebra are traceless, except for the lowest rank $r = 0$, which corresponds to $\mathbb{1}$, and the highest rank matrix with $r = D$, which is traceless only for even D. Let us prove this last statement.

Proof. First we need to show that the higher rank γ-matrices can be expressed as the alternate commutators or anti-commutators

$$\begin{aligned} \gamma^{\mu\nu} =& \frac{1}{2}[\gamma^\mu, \gamma^\nu], \\ \gamma^{\mu_1\mu_2\mu_3} =& \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\}, \\ \gamma^{\mu_1\mu_2\mu_3\mu_4} =& \frac{1}{2}[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}], \\ &\text{etc.} \end{aligned} \tag{D.1.5}$$

The first identity is trivial, since by definition $\gamma^{\mu\nu}$ includes a commutator. For the second identity, notice that

$$\frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\} = \frac{1}{4}\left\{\gamma^{\mu_1}, \gamma^{\mu_2}\gamma^{\mu_3} - \gamma^{\mu_3}\gamma^{\mu_2}\right\} = \frac{1}{4}\left\{\gamma^{\mu_1}, \gamma^{\mu_2}\gamma^{\mu_3}\right\} - \frac{1}{4}\left\{\gamma^{\mu_1}, \gamma^{\mu_3}\gamma^{\mu_2}\right\},$$

where we need to assume $\mu_2 \neq \mu_3$ in order to avoid a trivial result. We see that we also need to assume $\mu_1 \neq \mu_2$ and $\mu_1 \neq \mu_3$, because otherwise we would get a zero anti-commutator value. Thus

$$\begin{aligned} \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\} &= \frac{1}{4}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} + \gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_1} - \gamma^{\mu_1}\underbrace{\gamma^{\mu_3}\gamma^{\mu_2}}_{-\gamma^{\mu_2}\gamma^{\mu_3}} - \underbrace{\gamma^{\mu_3}\gamma^{\mu_2}}_{-\gamma^{\mu_2}\gamma^{\mu_3}}\gamma^{\mu_1}) = \\ &= \frac{1}{4}(2\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} + 2\underbrace{\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_1}}_{\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}}) = \gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} = \gamma^{\mu_1\mu_2\mu_3}. \end{aligned}$$

In general, we can write the identity

$$\gamma^{\mu_1\cdots\mu_D} = \frac{1}{2}\left(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D} - (-)^D\gamma^{\mu_2\cdots\mu_D}\gamma^{\mu_1}\right),$$

which covers all the different cases of (D.1.5), as

$$\begin{aligned} &\text{For} \quad D = 2m \quad \rightarrow \quad (-)^D = + \rightarrow \gamma^{\mu_1\cdots\mu_D} = \frac{1}{2}[\gamma^{\mu_1}, \gamma^{\mu_2\cdots\mu_D}], \\ &\text{For} \quad D = 2m+1 \quad \rightarrow \quad (-)^D = - \quad \rightarrow \gamma^{\mu_1\cdots\mu_D} = \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\cdots\mu_D}\right\}. \end{aligned}$$

Now, using the linearity and the cyclic property of the trace, we see that, for even dimension

$$\mathrm{Tr}(\gamma^{\mu_1\cdots\mu_D}) = \frac{1}{2}\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D}) - \frac{1}{2}\underbrace{\mathrm{Tr}(\gamma^{\mu_2\cdots\mu_D}\gamma^{\mu_1})}_{\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D})} = 0. \tag{D.1.6}$$

□

The next step is to guess the dimension of the Clifford algebra, i.e. the number of independent elements, for even dimension. Notice that there are C_r^D independent index choices at rank r (it is a binomial number because there are C_r^D different ways of choosing sets of r elements out of a set of D elements). Therefore, by virtue of the Binomial Theorem

$$\sum_{k=0}^{n} C_k^n r^k = \sum_{k=0}^{n} \binom{n}{k} r^k = (1+r)^n\,, \tag{D.1.7}$$

and taking into account that for $D = 2m$ all the γ-matrices are linearly independent, we see that the dimension of the Clifford algebra for even dimension is

$$\sum_{r=0}^{D} C_r^D = 2^D. \tag{D.1.8}$$

Useful relations involving γ-matrices

Here we show some tricks to multiply γ-matrices, that we will be using later. For example, it is worth remembering the general order reversal symmetry rule

$$\gamma^{\nu_1\ldots\nu_r} = \gamma^{\nu_1}\cdots\gamma^{\nu_r} = (-)^{\frac{r(r-1)}{2}}\gamma^{\nu_r}\cdots\gamma^{\nu_1} = (-)^{\frac{r(r-1)}{2}}\gamma^{\nu_r\ldots\nu_1}. \tag{D.1.9}$$

We have used the anti-commutativity of the γ-matrices a number $C_2^r = \frac{r(r-1)}{2}$ of times. Another interesting contraction is

$$\gamma_\nu\gamma^\mu\gamma^\nu = \gamma_\nu\left(2\eta^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu\right) = (2-D)\gamma^\mu. \tag{D.1.10}$$

Proceeding in the same way,

$$\begin{aligned} \gamma_\rho\gamma^{\mu\nu}\gamma^\rho &= \gamma_\rho\gamma^\mu\gamma^\nu\gamma^\rho = \gamma_\rho\gamma^\mu(2\eta^{\nu\rho}\mathbb{1} - \gamma^\rho\gamma^\nu) = 2\gamma^\nu\gamma^\mu - \gamma_\rho\left(2\eta^{\mu\rho}\mathbb{1} - \gamma^\rho\gamma^\mu\right)\gamma^\nu \\ &= -2\gamma^\mu\gamma^\nu - 2\gamma^\mu\gamma^\nu + \gamma_\rho\gamma^\rho\gamma^\mu\gamma^\nu = (D-4)\gamma^{\mu\nu}. \end{aligned} \tag{D.1.11}$$

In general, one has

$$\gamma_\rho\gamma^{\mu_1\ldots\mu_r}\gamma^\rho = (-)^r(D-2r)\gamma^{\mu_1\ldots\mu_r}. \tag{D.1.12}$$

D.1.1 Basis for even dimension

Restricting to even dimension $D = 2m$, an orthogonal basis of the Clifford algebra can be denoted by the following array $\left\{\Gamma^A\right\}$ of matrices

$$\left\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1\ldots\mu_D}\right\}. \tag{D.1.13}$$

Index values satisfy $\mu_1 < \mu_2 < ... < \mu_r$. It will be convenient to define a basis $\{\Gamma_A\}$ as the one containing the same elements as $\left\{\Gamma^A\right\}$ but in reverse order:

$$\{\Gamma_A = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \cdots, \gamma_{\mu_D\ldots\mu_1}\}. \tag{D.1.14}$$

Next we derive the trace orthogonality property. If we consider the product $\Gamma^A\Gamma_B$, we see that for different elements $A \neq B$ we get another arbitrary element of the Clifford algebra different from $\mathbb{1}$, which is traceless, as we previously discussed. For the same elements $A = B$, we always end up with the identity $\mathbb{1}$ (without sign changes involved because of the way we have defined Γ_B), whose trace is 2^m. This can be summarized as

$$\mathrm{Tr}(\Gamma^A\Gamma_B) = 2^m\delta^A_B. \tag{D.1.15}$$

The list in (D.1.13) contains 2^D trace orthogonal matrices, which is the same as the number of elements of a matrix M of dimension $2^m \times 2^m$. Therefore $\left\{\Gamma^A\right\}$ constitutes a basis of the space of matrices M of dimension $2^m \times 2^m$, and we can write the expansion

$$M = \sum_A m_A\Gamma^A. \tag{D.1.16}$$

It is easy to obtain the coefficients m_A of the expansion with the help of the orthogonality property

$$\frac{1}{2^m}\mathrm{Tr}(M\Gamma_A) = \frac{1}{2^m}\mathrm{Tr}\left(\sum_B m_B\Gamma^B\Gamma_A\right) = \frac{1}{2^m}\sum_B m_B\mathrm{Tr}\left(\Gamma^B\Gamma_A\right) = m_A. \tag{D.1.17}$$

For odd dimensions, $D = 2m + 1$, the situation is somewhat different, because not all the γ-matrices are independent, as we discussed before. In fact, a basis of the Clifford algebra for odd dimension only contains the matrices in (D.1.13) up to rank m. For a further discussion, see [30].

D.1.2 The highest rank Clifford algebra element

The highest rank element of the Clifford algebra has special importance in physics, as it is deeply related to the chirality of fermions. We define the quantitity

$$\gamma_* \equiv (-i)^{m+1}\gamma_0\gamma_1 \ldots \gamma_{D-1}, \tag{D.1.18}$$

which satisfies $\gamma_*^2 = \mathbb{1}$ in every even dimension, and it is Hermitian $\gamma_*^\dagger = \gamma_*$. We proceed to prove those properties.

Proof. For the first property

$$\gamma_*^2 = \underbrace{(-i)^{2m+2}}_{-(-)^{D/2}} \gamma_0\gamma_1 ... \gamma_{D-1}\gamma_0\gamma_1 ... \gamma_{D-1}$$
$$= (-)^{\frac{D}{2}+1}\gamma_0\gamma_1...\gamma_{D-1}\gamma_{D-1}...\gamma_1\gamma_0(-)^{\frac{D(D-1)}{2}} = (-)^{\frac{D^2}{2}+1}\gamma_0^2 = \mathbb{1}, \tag{D.1.19}$$

whereas for the Hermiticity property

$$\gamma_*^\dagger = [(-i)^{m+1}]^\dagger \underbrace{\gamma_{D-1}^\dagger ... \gamma_1^\dagger\gamma_0^\dagger}_{-\gamma_{D-1}...\gamma_0} = -i^{\frac{D}{2}+1}(-)^{\frac{D(D-1)}{2}}\gamma_0\gamma_1...\gamma_{D-1}$$
$$= \underbrace{(-)^{\frac{D^2}{2}}(-)^{\frac{D}{2}-1}i^{\frac{D}{2}+1}}_{(-)^{m+1}i^{m+1}}\gamma_0\gamma_1...\gamma_{D-1} = \gamma_*. \tag{D.1.20}$$

□

For dimension $D = 4$ the matrix γ_* is typically called γ_5. The matrix γ_* is related to the unique highest rank element in (D.1.13) by

$$i^{m+1}\varepsilon_{\mu_1\mu_2...\mu_D}\gamma_* = \varepsilon_{\mu_1\mu_2...\mu_D}\gamma_0\gamma_1...\gamma_{D-1} = \varepsilon_{\mu_1\mu_2...\mu_D}\gamma_{01...D-1} = \gamma_{\mu_1\mu_2...\mu_D}, \tag{D.1.21}$$

where the Levi-Civita tensor reproduces the complete antisymmetry of $\gamma_{\mu_1\mu_2...\mu_D}$. Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\gamma_* = 0$, it follows that one can find a representation such that

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{D.1.22}$$

With this, we see that the Weyl fields ψ and χ can be obtained from a Dirac field Ψ by applying the chiral projectors:

$$P_L = \frac{1}{2}(\mathbb{1} + \gamma_*) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \frac{1}{2}(\mathbb{1} - \gamma_*) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{D.1.23}$$

Therefore,

$$\begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \equiv P_L\Psi, \qquad \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \equiv P_R\Psi. \tag{D.1.24}$$

Matrices in (D.1.23) indeed project to orthogonal subspaces, since they satisfy $P_LP_L = P_L$, $P_RP_R = P_R$ and $P_LP_R = 0$.

We are going to show that both the anti-commutator $\{\gamma_*, \gamma^\mu\}$ and the commutator $[\gamma_*, \gamma^{\mu\nu}]$ are zero. This result will prove to be useful soon. We compute the anti-commutator, taking into account that γ^μ anticommutes with all matrices $\gamma_0, \gamma_1...\gamma_{D-1}$ except with one, which has the same index value μ

$$\{\gamma_*, \gamma^\mu\} = (-i)^{m+1}\left(\gamma_0\gamma_1...\gamma_{D-1}\gamma^\mu + \gamma^\mu\gamma_0\gamma_1...\gamma_{D-1}\right)$$
$$= (-i)^{m+1}\left((-)^{D-1}\gamma^\mu\gamma_0\gamma_1...\gamma_{D-1} + \gamma^\mu\gamma_0\gamma_1...\gamma_{D-1}\right) = 0. \tag{D.1.25}$$

Repeating the same procedure for the commutator

$$\begin{aligned}[\gamma_*, \gamma^{\mu\nu}] &= (-i)^{m+1}\left(\gamma_0\gamma_1...\gamma_{D-1}\gamma^{\mu\nu} - \gamma^{\mu\nu}\gamma_0\gamma_1...\gamma_{D-1}\right)\\ &= (-i)^{m+1}\Big((-)^{2D-2}\gamma^{\mu\nu}\gamma_0\gamma_1...\gamma_{D-1} - \gamma^{\mu\nu}\gamma_0\gamma_1...\gamma_{D-1}\Big) = 0. \end{aligned} \tag{D.1.26}$$

We next consider a general block form

$$\gamma^\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{D.1.27}$$

for the γ-matrices in a representation where (D.1.22) holds. Because of (D.1.25) we have $\gamma_*\gamma^\mu = -\gamma^\mu\gamma_*$ or, in matrix terms,

$$\begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = -\begin{pmatrix} A & -B \\ C & -D \end{pmatrix}.$$

This means that $A = -A$ and $D = -D$ and thus $A = D = 0$. We have arrived at the important result that the γ^μ can be given in a block-off diagonal form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{D.1.28}$$

The matrices σ^μ and $\bar{\sigma}^\mu$ can be thought of as $2^{m-1} \times 2^{m-1}$ generalizations of the Pauli matrices. In an analogous way, by using (D.1.26) it can be shown that the matrices $\gamma^{\mu\nu}$ take the block diagonal form

$$\gamma^{\mu\nu} = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}. \tag{D.1.29}$$

D.1.3 The charge conjugation matrix

The charge conjugation matrix, C, is defined as the unitary matrix satisfying that each matrix $C\Gamma^A$ is either symmetric or antisymmetric. Symmetry depends only on the rank r of the matrix Γ^A, so we can write:

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)}, \qquad t_r = \pm 1, \tag{D.1.30}$$

being $\Gamma^{(r)}$ a rank r matrix in the basis (D.1.13) [2]. For rank $r = 0$ and 1, one obtains

$$C^T = -t_0 C, \qquad \gamma^{\mu T} = t_0 t_1 C\gamma^\mu C^{-1} \tag{D.1.31}$$

We have used that $t_r = 1/t_r$ and that $(AB)^T = B^T A^T$ for any two matrices A and B. These relations are enough to determine the symmetries of all $C\gamma^{\mu_1 \cdots \mu_r}$ and thus all coefficients t_r. For example,

$$\begin{aligned}(C\gamma^{\mu_1\mu_2})^T = \gamma^{\mu_1\mu_2 T}C^T = -t_0\gamma^{\mu_1\mu_2 T}C = -t_0\gamma^{\mu_2 T}\gamma^{\mu_1 T}C = -t_2 C\gamma^{\mu_1}\gamma^{\mu_2},\\ t_0(t_0t_1)(t_0t_1)C\gamma^{\mu_2}C^{-1}C\gamma^{\mu_1}C^{-1}C = t_2 C\gamma^{\mu_1}\gamma^{\mu_2},\\ t_0 C\gamma^{\mu_2\mu_1} = t_2 C\gamma^{\mu_1\mu_2} \;\to\; \boxed{t_2 = -t_0}. \end{aligned} \tag{D.1.32}$$

[2]The $-$ sign in (D.1.30) is introduced just for convenience in the calculations.

In a similar way one can show that $t_3 = -t_1$. In general, $t_{r+4} = t_r$. The following matrices are valid charge conjugation matrices for even dimension,

$$C_+ = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes ..., \text{ for the cases in which } t_0 t_1 = 1, \tag{D.1.33}$$

$$C_- = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes ..., \text{ for the cases in which } t_0 t_1 = -1. \tag{D.1.34}$$

We can check for example that C_+ satisfies (D.1.31). Taking into account that $(A \otimes B)^T = A^T \otimes B^T$ for any two matrices A and B, we see that

$$C_+^T = \sigma_1 \otimes (-\sigma_2) \otimes \sigma_1 \otimes (-\sigma_2) \otimes ... \tag{D.1.35}$$

So we have indeed $C^T = -t_0 C$, and this shows clearly that t_0 depends on the dimension. Now we choose γ^3 from (D.1.1), and make use of the fact that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

$$\begin{aligned} C_+ \gamma^3 C_+^{-1} &= \left(\sigma_1 \sigma_3 \sigma_1^{-1} \otimes \sigma_2 \sigma_1 \sigma_2^{-1} \otimes \sigma_1 \mathbb{1} \sigma_1^{-1} \otimes ...\right) = \left(-\sigma_1 \sigma_3 \sigma_1^{-1} \otimes -\sigma_2 \sigma_1 \sigma_2^{-1} \otimes \mathbb{1} \otimes ...\right) \\ &= (\sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ...) = \left(\sigma_3^T \otimes \sigma_1^T \otimes \mathbb{1} \otimes ...\right) = (\gamma^3)^T \end{aligned} \tag{D.1.36}$$

The values of t_0 and t_1 (and thus all t_r) depend on the spacetime dimension D mod8 and on the rank r mod4. That is, their values are repeated every eight dimensions D and every four ranks r. The later can be seen from $t_{t+4} = t_r$. The former can be seen by looking the matrices $C_\pm$. For example, let us pick $D = 2$. In this case, $t_0 = \pm 1$ for $C_\mp$. If we start increasing the dimension, we won't get $t_0 = \pm 1$ for $C_\mp$ again until $D = 10$.

In the Table D.1.3 we give the values r mod4 for which $t_r = \pm 1$, for each D mod8. Notice that, as $t_2 = -t_0$ and $t_3 = -t_1$, we will always have a pair of values corresponding to $t_r = -1$ and another pair corresponding to $t_r = +1$. These entries in the table are determined by counting the number of symmetric and antisymmetric matrices in every dimension. For even dimension C_+ and C_- are possible choices. For odd dimension, C is unique. In fact, it is either C_+ or C_- (see [41]). This explains why in the Table there are two possible choices for even dimension. The Table D.1.3 is fundamental to explain why Majorana spinors can only exist in certain dimensions.

The symmetry property of a γ-matrix fixes also its complex conjugation property. To see this, we define

$$B \equiv i t_0 C \gamma^0, \tag{D.1.37}$$

which is unitary, as $B^{-1} = i^{-1} t_0 (\gamma^0)^{-1} C^{-1} = i t_0 \gamma^0 C^\dagger = B^\dagger$. Important identities involving B are

$$\gamma^{\mu *} = -t_0 t_1 B \gamma^\mu B^{-1}, \qquad B^* B = -t_1 \mathbb{1}. \tag{D.1.38}$$

Proof. By taking the complex conjugate of (D.1.31) we have $\gamma^{\mu\dagger} = t_0 t_1 C^* \gamma^{\mu *} C^{-1*}$. We solve for $\gamma^{\mu *}$ using that $C^* = -t_0 C^{-1}$ and $C^{-1*} = -t_0 C$

$$\gamma^{\mu *} = t_0 t_1 C \gamma^{\mu\dagger} C^{-1} = t_0 t_1 (-i t_0)(i t_0) C \gamma^0 \gamma^\mu \gamma^0 C^{-1} = -t_0 t_1 B \gamma^\mu B^\dagger = -t_0 t_1 B \gamma^\mu B^{-1}$$

D (mod8)	$t_r = -1$	$t_r = +1$
0	0, 3	2, 1
	0, 1	2, 3
1	0, 1	2, 3
2	0, 1	2, 3
	1, 2	0, 3
3	1, 2	0, 3
4	1, 2	0, 3
	2, 3	0, 1
5	2, 3	0, 1
6	2, 3	0, 1
	0, 3	1, 2
7	0, 3	1, 2

Table D.1: *Symmetries of γ-matrices. The entries contain the matrix ranks r mod4 for which $t_r = \pm 1$, corresponding to each spacetime dimension D mod8.*

Now, in order to prove $B^* B = -t_1 \mathbb{1}$, we first compute B^*

$$B^* = -it_0 C^* (\gamma^0)^* = it_0 C^* \gamma^0 = -it_0^2 C^{-1} \gamma^0 = -iC^{-1}\gamma^0.$$

In this way

$$B^* B = t_0 C^{-1} \gamma^0 C \gamma^0 = t_0^2 t_1 (\gamma^0)^T \gamma^0 = t_1 (\gamma^0)^2 = -t_1 \mathbb{1}.$$

□

D.2 Spinors in arbitrary dimension

D.2.1 Spinor bilinears

As it is discussed in the section 2.2 , spinor components are anti-commuting Grassmann numbers

$$\Psi_\alpha \Psi_\beta = -\Psi_\beta \Psi_\alpha. \tag{D.2.1}$$

Throughout the rest of this appendix, when writing a bar ove a spinor, we are assuming the Majorana conjugate (2.3.1), not the Dirac adjoint. Consider two different arbitrary spinors $\boldsymbol{\chi}$ and $\boldsymbol{\lambda}$. We can build a general bilinear form using matrices from the Clifford algebra:

$$\bar{\boldsymbol{\lambda}} \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\chi}. \tag{D.2.2}$$

This is indeed a bilinear form, i.e. it takes two "vector-like" quantities $\boldsymbol{\lambda}$ and $\boldsymbol{\chi}$ and returns a scalar value. We can express it in a different way by making use of (D.1.30)

$$\begin{aligned} \bar{\boldsymbol{\lambda}} \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\chi} &= \boldsymbol{\lambda}^T C \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\chi} = -t_r \boldsymbol{\lambda}^T \gamma^T_{\mu_1 \dots \mu_r} C^T \boldsymbol{\chi} \\ &= (-) - t_r \left(\boldsymbol{\chi}^T C \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\lambda} \right)^T = t_r \bar{\boldsymbol{\chi}} \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\lambda}. \end{aligned} \tag{D.2.3}$$

In the last step we have used that the transpose operation doesn't affect a scalar quantity (a extra minus sign is coming from (D.2.1)). The term *Majorana flip relation* is used to refer to (D.2.3).

The previous relation also mean the following rule. For any relation between spinors including γ-matrices, there is a corresponding relation between the barred spinors. To see this more clear, note that, by (D.2.3)

$$\bar{\boldsymbol{\chi}}_1 \boldsymbol{\chi}_2 = t_0 \bar{\boldsymbol{\chi}}_2 \boldsymbol{\chi}_1 \tag{D.2.4}$$

Then, if we have the relation between spinors $\boldsymbol{\chi}_2 = \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\lambda}$, because of (D.2.4) we see there is a corresponding relation between barred spinors

$$\bar{\boldsymbol{\chi}}_1 \boldsymbol{\chi}_2 = t_0 \bar{\boldsymbol{\chi}}_2 \boldsymbol{\chi}_1 = \bar{\boldsymbol{\chi}}_1 \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\lambda} = t_r \bar{\boldsymbol{\lambda}} \gamma_{\mu_1 \dots \mu_r} \boldsymbol{\chi}_1 \ \rightarrow \ \bar{\boldsymbol{\chi}}_2 = t_0 t_r \bar{\boldsymbol{\lambda}} \gamma_{\mu_1 \dots \mu_r}. \tag{D.2.5}$$

Using the spin part of an infinitesimal Lorentz transformation $\delta \boldsymbol{\chi} = -\frac{1}{4} \lambda^{\mu\nu} \gamma_{\mu\nu} \boldsymbol{\chi}$, we can prove that the spinor bilinear $\bar{\boldsymbol{\lambda}} \boldsymbol{\chi}$ is a Lorentz scalar.

Proof. We simply compute the transformation, taking (D.2.5) into account for the expression of $\overline{\delta \boldsymbol{\lambda}}$:

$$\begin{aligned} \delta(\bar{\boldsymbol{\lambda}} \boldsymbol{\chi}) =& \delta \bar{\boldsymbol{\lambda}} \boldsymbol{\chi} + \bar{\boldsymbol{\lambda}} \delta \boldsymbol{\chi} = \overline{\delta \boldsymbol{\lambda}} \boldsymbol{\chi} + \bar{\boldsymbol{\lambda}} \delta \boldsymbol{\chi} = -\frac{1}{4} \lambda^{\mu\nu} \overline{\gamma_{\mu\nu} \boldsymbol{\lambda}} \boldsymbol{\chi} - \frac{\bar{\boldsymbol{\lambda}}}{4} \lambda^{\mu\nu} \gamma_{\mu\nu} \boldsymbol{\chi} = \\ =& -\frac{1}{4} \lambda^{\mu\nu} t_0 t_2 \bar{\boldsymbol{\lambda}} \gamma_{\mu\nu} \boldsymbol{\chi} - \frac{\bar{\boldsymbol{\lambda}}}{4} \lambda^{\mu\nu} \gamma_{\mu\nu} \boldsymbol{\chi} = +\frac{\bar{\boldsymbol{\lambda}}}{4} \lambda^{\mu\nu} \gamma_{\mu\nu} \boldsymbol{\chi} - \frac{\bar{\boldsymbol{\lambda}}}{4} \lambda^{\mu\nu} \gamma_{\mu\nu} \boldsymbol{\chi} = 0. \end{aligned}$$

□

D.2.2 Spinor indices

Although frequently omitted, spinor indices are sometimes necessary. Tipically, the components of basic spinors $\boldsymbol{\lambda}$ are indicated as λ_α and the components of barred spinors $\bar{\boldsymbol{\lambda}}$ as λ^α. We introduce a matrix to raise indices such that

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta} \lambda_\beta. \tag{D.2.6}$$

Since $\bar{\boldsymbol{\lambda}}^T = C^T \boldsymbol{\lambda}$, we note that $\mathcal{C}^{\alpha\beta}$ are the components of C^T. We can also introduce a lowering matrix

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}. \tag{D.2.7}$$

In order for these two equations to be mutually consistent,

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha} = \mathcal{C}^{\beta\gamma} \lambda_\gamma \mathcal{C}_{\beta\alpha}, \tag{D.2.8}$$

we must impose

$$\mathcal{C}^{\alpha\beta} \mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha, \qquad \mathcal{C}_{\beta\alpha} \mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma. \tag{D.2.9}$$

When operating with γ-matrices, spinor indices are written as $(\gamma_\mu)_\alpha{}^\beta$. Their indices can also be raised or lowered using the charge conjugation matrix. For instance

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\sigma \mathcal{C}_{\sigma\beta}. \tag{D.2.10}$$

From (D.1.30), we see that γ-matrices with all spinor indices upstairs or downstairs are either symmetric or antisymmetric

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r(\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha}. \tag{D.2.11}$$

It is intriguing that raising and lowering indices can produce a minus sign depending on the dimension, as opposed to what happened with spacetime indices. In fact,

$$\lambda^\alpha\chi_\alpha = -t_0 C^{\alpha\beta}\lambda_\beta\chi^\gamma(C^{-1})_{\gamma\alpha} = -t_0\delta_\gamma{}^\beta\lambda_\beta\chi^\gamma = -t_0\lambda_\alpha\chi^\alpha. \tag{D.2.12}$$

In $D = 4$ for example we have $\lambda^\alpha\chi_\alpha = -\lambda_\alpha\chi^\alpha$.

D.2.3 Fierz reordering

Fierz reordering is a technique that exploits the fact that the Clifford set $\left\{\Gamma^A\right\}$ forms a complete basis of any $2^m \times 2^m$ matrix M, in order to obtain expressions involving the products of spinor bilinears, called *Fierz identities*. These identities are important in SUSY theories.

We derive now the basic Fierz identity. We take $\delta_\alpha{}^\beta\delta_\gamma{}^\delta$, which can be considered as a matrix in the indices γ and β with the indices α and δ having the function of labelling different matrices. We apply (D.1.16) to this matrix

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \sum_A (m_A)_\alpha^\delta(\Gamma^A)_\gamma{}^\beta. \tag{D.2.13}$$

The coefficients are $(m_A)_\alpha^\delta = 2^{-m}\mathrm{Tr}(\delta_\alpha{}^\tau\delta_\rho{}^\delta(\Gamma_A)_\tau{}^\sigma) = 2^{-m}\delta_\alpha{}^\tau\delta_\rho{}^\delta(\Gamma_A)_\tau{}^\rho$. Inserting this in (D.2.13) we get

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \frac{1}{2^m}\sum_A \delta_\alpha{}^\tau(\Gamma_A)_\tau{}^\rho\delta_\rho{}^\delta(\Gamma^A)_\gamma{}^\beta, \tag{D.2.14}$$

or, equivalently

$$\boxed{\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \frac{1}{2^m}\sum_A (\Gamma_A)_\alpha{}^\delta(\Gamma^A)_\gamma{}^\beta}. \tag{D.2.15}$$

This is the basic Fierz identity. The next point is to derive an important identity which is needed for SUSY Yang-Mills theories. Instead of the matrix $\delta_\alpha{}^\beta\delta_\gamma{}^\delta$, we consider the matrix $(\gamma^\mu)_\alpha{}^\beta(\gamma_\mu)_\gamma{}^\delta$, with the indices α and δ playing again the role of labelling different matrices. Proceeding in the same way as before we get

$$(\gamma^\mu)_\alpha{}^\beta(\gamma_\mu)_\gamma{}^\delta = \frac{1}{2^m}\sum_A (\gamma^\mu)_\alpha{}^\tau(\Gamma_A)_\tau{}^\rho(\gamma_\mu)_\rho{}^\delta(\Gamma^A)_\gamma{}^\beta. \tag{D.2.16}$$

Taking into account the result (D.1.12), we can express (D.2.16) as

$$(\gamma^\mu)_\alpha{}^\beta(\gamma_\mu)_\gamma{}^\delta = \frac{1}{2^m}\sum_A(-)^{r_A}(D-2r_A)(\Gamma_A)_\alpha{}^\delta(\Gamma^A)_\gamma{}^\beta, \tag{D.2.17}$$

where r_A is the rank of the element Γ_A. Now we lower the indices β and δ and we consider the fully symmetric part in $(\beta\gamma\delta)$:

$$(\gamma^\mu)_{\alpha(\beta}(\gamma_\mu)_{\gamma\delta)} = \frac{1}{2^m}\sum_A(-)^{r_A}(D-2r_A)(\Gamma_A)_{\alpha(\delta}(\Gamma^A)_{\gamma\beta)}. \tag{D.2.18}$$

Writing the indices in this way we can use the symmetry/antisymmetry property (D.2.11), taking the form $(\gamma_\mu)_{\alpha\beta} = -t_1(\gamma_\mu)_{\beta\alpha}$. If we expand $(\gamma^\mu)_{\alpha(\beta}(\gamma_\mu)_{\gamma\delta)}$ in its six terms, we see all of them cancel by pairs if the γ-matrices are antisymmetric $(\gamma_\mu)_{\alpha\beta} = -(\gamma_\mu)_{\beta\alpha}$. Thus, $(\gamma^\mu)_{\alpha(\beta}(\gamma_\mu)_{\gamma\delta)}$ does not vanish only for the dimensions in which $t_1 = -1$. By checking Table D.1.3, we see this only happens for $D = 3, 4$. Let us restrict to $D = 4$. From the right-hand side of (D.2.18), we see that for $D = 4$ the term $(D - 2r_A)$ is only non-vanishing for $r_A = 1$. Thus only rank 1 matrices contribute to the right-hand side, and so we can write (D.2.18) as

$$(\gamma^\mu)_{\alpha(\beta}(\gamma_\mu)_{\gamma\delta)} = -\frac{1}{2}(\gamma_\mu)_{\alpha(\delta}(\gamma^\mu)_{\gamma\beta)}, \tag{D.2.19}$$

or using the symmetry property

$$(\gamma^\mu)_{\alpha(\beta}(\gamma_\mu)_{\gamma\delta)} = 0. \tag{D.2.20}$$

Finally, if we multiply this equation with three spinors $\boldsymbol{\lambda}_1^\beta$, $\boldsymbol{\lambda}_2^\gamma$ and $\boldsymbol{\lambda}_3^\delta$, we can write (D.2.20) as

$$\boxed{\gamma_\mu\boldsymbol{\lambda}_{[1}\bar{\boldsymbol{\lambda}}_2\gamma^\mu\boldsymbol{\lambda}_{3]} = 0}, \tag{D.2.21}$$

where the symmetry of the indices in (D.2.20) has become antisymmetry among the three spinors because of their anti-commutativity property.

D.2.4 Charge conjugation of spinors

Charge conjugation is an operation that acts on spinors, analogous to complex conjugation. It is certainly possible to apply also complex conjugation to spinors, but it turns out to be much easier to consider the operation of charge conjugation, which acts in the same way for scalar quantities. The charge conjugate of any spinor λ is defined as

$$\boldsymbol{\lambda}^C \equiv B^{-1}\boldsymbol{\lambda}^*, \tag{D.2.22}$$

where B is the matrix defined in (D.1.37).

The charge conjugate of a $2^m \times 2^m$ matrix M is defined as

$$M^C \equiv B^{-1}M^*B. \tag{D.2.23}$$

In the same way as complex conjugation, charge conjugation does not affect the order of the matrices: $(MN)^C = B^{-1}M^*N^*B = M^C N^C$. The charge conjugate of a γ-matrix is actually very simple

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0 t_1)\gamma_\mu. \tag{D.2.24}$$

The rule for the complex conjugate of a spinor bilinear with an arbitrary matrix M is

$$(\bar{\boldsymbol{\chi}} M \boldsymbol{\lambda})^* \equiv (\bar{\boldsymbol{\chi}} M \boldsymbol{\lambda})^C = (-t_0 t_1)\overline{\boldsymbol{\chi}^C} M^C \boldsymbol{\lambda}^C. \tag{D.2.25}$$

Proof. We compute the following

$$(-t_0 t_1)\overline{\boldsymbol{\chi}^C} M^C \boldsymbol{\lambda}^C = (-t_0 t_1) = (-t_0 t_1)(\boldsymbol{\chi}^C)^T C B^{-1} M^* B B^{-1} \boldsymbol{\lambda}^*.$$

On the other hand, $CB^{-1} = -it_0 C\gamma_0 C^{-1} = -it_1\gamma_0$, as $\gamma_0^T = \gamma_0$, and it is also true that $(B^{-1})^T = B^* = it_0 C^* \gamma_0$. We introduce these relations in the previous equation:

$$\begin{aligned}(-t_0 t_1)\overline{\boldsymbol{\chi}^C} M^C \boldsymbol{\lambda}^C &= (-t_0 t_1)(B^{-1}\boldsymbol{\chi}^*)^T C B^{-1} M^* \boldsymbol{\lambda}^* = it_0(\boldsymbol{\chi}^*)^T (B^{-1})^T \gamma_0 M^* \boldsymbol{\lambda}^* \\ &= (\boldsymbol{\chi}^*)^T C^* M^* \boldsymbol{\lambda}^* = (\bar{\boldsymbol{\chi}} M \boldsymbol{\lambda})^* = (\bar{\boldsymbol{\chi}} M \boldsymbol{\lambda})^C. \end{aligned} \tag{D.2.26}$$

□

Any spinor $\boldsymbol{\lambda}$ and its conjugate $\boldsymbol{\lambda}^C$ transform in the same way under a Lorentz transformation. We proceed to derive this important fact.

Proof. The spin part of an infinitesimal transformation of any spinor $\boldsymbol{\lambda}$ is $\delta\boldsymbol{\lambda} = -\frac{1}{4}\lambda^{\mu\nu}\gamma_{\mu\nu}\boldsymbol{\lambda}$. Now, taking the charge conjugate of this equation we have

$$\delta\boldsymbol{\lambda}^C = -\frac{\lambda^{\mu\nu}}{4}(\gamma_{\mu\nu}\boldsymbol{\lambda})^C = -\frac{\lambda^{\mu\nu}}{4}B^{-1}(\gamma_{\mu\nu}\boldsymbol{\lambda})^* = -\frac{\lambda^{\mu\nu}}{4}B^{-1}\gamma_{\mu\nu}^* B B^{-1}\boldsymbol{\lambda}^* = -\frac{\lambda^{\mu\nu}}{4}\gamma_{\mu\nu}^C \boldsymbol{\lambda}^C,$$

where we have used the definitions (D.2.22) and (D.2.23). But taking into account (D.2.24), notice that, for $\mu \neq \nu$

$$\gamma_{\mu\nu}^C = (\gamma_\mu\gamma_\nu)^C = \gamma_\mu^C \gamma_\nu^C = (-t_0 t_1)^2 \gamma_\mu\gamma_\nu = \gamma_{\mu\nu}.$$

(for $\mu = \nu$, $\gamma_{\mu\nu} = \pm\mathbb{1}$ and it is clear that it is its own conjugate). Thus $\delta\boldsymbol{\lambda}^C = -\frac{\lambda^{\mu\nu}}{4}\gamma_{\mu\nu}\boldsymbol{\lambda}^C$, i.e. the conjugate $\boldsymbol{\lambda}^C$ transforms in the same way as $\boldsymbol{\lambda}$. □

Finally, we compute the charge conjugate of the highest rank Clifford algebra element γ_*:

$$(\gamma_*)^C = i^{m+1}\gamma_0^C\gamma_1^C \cdots \gamma_{D-1}^C = i^{m+1}\underbrace{(-t_0 t_1)^D}_{+1}\gamma_1\gamma_2\cdots\gamma_{D-1} = (-)^{D/2+1}\gamma_*. \tag{D.2.27}$$

Thus, for dimension $D = 4$ for example, one has $(\gamma_*)^C = -\gamma_*$, which means $(P_L)^C = P_R$.

Appendix E

Mathematica code

In this Appendix we show the Mathematica notebooks that have been used to obtain the specific expression for the generators of the Lorentz group/Clifford algebra, and also to check their commutation/anticommutation relations. Comments and explanations are included.

A basis of Γ-matrices in $D = 10$

We consider D=10. We use the general and explicit construction of γ-matrices.

In[2]:=
```
Gamma0 = i * KroneckerProduct[PauliMatrix[1], IdentityMatrix[2],
    IdentityMatrix[2], IdentityMatrix[2], IdentityMatrix[2]];
```

In[3]:=
```
Gamma1 = KroneckerProduct[PauliMatrix[2], IdentityMatrix[2],
   IdentityMatrix[2], IdentityMatrix[2], IdentityMatrix[2]];
```

In[4]:=
```
Gamma2 = KroneckerProduct[PauliMatrix[3], PauliMatrix[1],
   IdentityMatrix[2], IdentityMatrix[2], IdentityMatrix[2]];
```

In[5]:=
```
Gamma3 = KroneckerProduct[PauliMatrix[3], PauliMatrix[2],
   IdentityMatrix[2], IdentityMatrix[2], IdentityMatrix[2]];
```

In[6]:=
```
Gamma4 = KroneckerProduct[PauliMatrix[3], PauliMatrix[3],
   PauliMatrix[1], IdentityMatrix[2], IdentityMatrix[2]];
```

In[7]:=
```
Gamma5 = KroneckerProduct[PauliMatrix[3], PauliMatrix[3],
   PauliMatrix[2], IdentityMatrix[2], IdentityMatrix[2]];
```

In[8]:=
```
Gamma6 = KroneckerProduct[PauliMatrix[3], PauliMatrix[3],
   PauliMatrix[3], PauliMatrix[1], IdentityMatrix[2]];
```

In[9]:=
```
Gamma7 = KroneckerProduct[PauliMatrix[3], PauliMatrix[3],
   PauliMatrix[3], PauliMatrix[2], IdentityMatrix[2]];
```

In[10]:=
```
Gamma8 = KroneckerProduct[PauliMatrix[3], PauliMatrix[3],
   PauliMatrix[3], PauliMatrix[3], PauliMatrix[1]];
```

In[11]:=
```
Gamma9 = KroneckerProduct[PauliMatrix[3], PauliMatrix[3],

   PauliMatrix[3], PauliMatrix[3], PauliMatrix[2]];
```

We check the dimension of the matrices

In[14]:=
```
Dimensions[Gamma0]
Dimensions[Gamma1]
Dimensions[Gamma2]
Dimensions[Gamma3]
Dimensions[Gamma4]
Dimensions[Gamma9]
```

Out[14]= {32, 32}

Out[15]= {32, 32}

Out[16]= {32, 32}

Out[17]= {32, 32}

Out[18]= {32, 32}

Out[19]= {32, 32}

As an example, we just show the 32x32 matrix Γ^0

In[21]:=
```
Gamma0 // MatrixForm
```

Out[21]//MatrixForm=

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We define a function that groups all the matrices

In[22]:=
```
Gam[u_] := KroneckerDelta[u, 1] Gamma0 + KroneckerDelta[u, 2] Gamma1 +
  KroneckerDelta[u, 3] Gamma2 + KroneckerDelta[u, 4] Gamma3 +
  KroneckerDelta[u, 5] Gamma4 + KroneckerDelta[u, 6] Gamma5 +
  KroneckerDelta[u, 7] Gamma6 + KroneckerDelta[u, 8] Gamma7 +
  KroneckerDelta[u, 9] Gamma8 + KroneckerDelta[u, 10] Gamma9
```

Now we define the Minkowski metric:

In[23]:=
```
et = DiagonalMatrix[{-1, 1, 1, 1, 1, 1, 1, 1, 1, 1}];
```

We define a expression for the identity to be checked. If the identity is satisfied, this expression needs to be zero.

In[24]:=
```
Identit[mu_, nu_] := (Gam[mu].Gam[nu]) + (Gam[nu].Gam[mu]) -
  (2 et[[mu, nu]] IdentityMatrix[32])
```

In[25]:=
```
Bigequation = Table[Identit[mu, nu], {mu, 1, 10}, {nu, 1, 10}];
```

In the variable Bigequation we have written a 10x10 table, with each entry containing a 32x32 matrix. All these matrices should be zero. The command Tally[Flatten[Bigequation]] gives me the number of zeros appearing in this table, which should be 100x32x32=102400.

```
Tally[Flatten[Bigequation]]
```

```
{{0, 102400}}
```

The Lie algebra of the Lorentz group SO(3,1) in $D = 4$

We consider spacetime dimension D=4. Let's obtain the specific expression for the generators of the Lorentz group in the basic representation. First we define the Minkowski metric:

In[2]:=
```
et = DiagonalMatrix[{-1, 1, 1, 1}];
```

In[3]:=
```
et // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we define this representation for the generators of the Lorentz group:

In[5]:=
```
m[rho_, sigma_, mu_, nu_] := KroneckerDelta[mu, rho] et[[nu, sigma]] -
  KroneckerDelta[mu, sigma] et[[rho, nu]]
```

We define the commutator between two different generators

In[6]:=
```
comm[mm_, nn_, rr_, ss_] :=
  (Array[m, {1, 1, 4, 4}, {mm, nn, 1, 1}][[1, 1]].
     Array[m, {1, 1, 4, 4}, {rr, ss, 1, 1}][[1, 1]]) -
   (Array[m, {1, 1, 4, 4}, {rr, ss, 1, 1}][[1, 1]].
     Array[m, {1, 1, 4, 4}, {mm, nn, 1, 1}][[1, 1]]);
```

We now define the explicit theoretical expression for the commutator:

In[7]:=
```
res[mi_, ni_, ri_, si_] :=
  et[[ni, ri]] Array[m, {1, 1, 4, 4}, {mi, si, 1, 1}][[1, 1]] -
   et[[mi, ri]] Array[m, {1, 1, 4, 4}, {ni, si, 1, 1}][[1, 1]] -
   et[[ni, si]] Array[m, {1, 1, 4, 4}, {mi, ri, 1, 1}][[1, 1]] +
   et[[mi, si]] Array[m, {1, 1, 4, 4}, {ni, ri, 1, 1}][[1, 1]];
```

We check that, for the same set of indices values, both lead to the same result

In[8]:=
```
Manipulate[comm[ma, na, ra, sa] //
  MatrixForm, {ma, 1, 4, 1}, {na, 1, 4, 1}, {ra, 1, 4, 1}, {sa, 1, 4, 1}]
```

Out[8]=

ma
na
ra
sa

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[9]:=
```
Manipulate[res[ma, na, ra, sa] //
  MatrixForm, {ma, 1, 4, 1}, {na, 1, 4, 1}, {ra, 1, 4, 1}, {sa, 1, 4, 1}]
```

Out[9]=

ma
na
ra
sa

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To check all the generators simultaneously, we impose define the equation eq = res – comm, which has 4x4x4x4=256 matrices of size 4x4. In total, this variable has 256x4x4=4096 components

In[16]:=
```
eq = Table[res[ma, na, ra, sa] - comm[ma, na, ra, sa],
   {ma, 4}, {na, 4}, {ra, 4}, {sa, 4}];
```

We check that all its components are actually zero

In[17]:=
```
Tally[Flatten[eq]]
```

Out[17]=
```
{{0, 4096}}
```

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Statement of Originality

D. GUILLERMO FRANCO ABELLÁN, estudiante del Grado en Física de la Facultad de Química de la Universidad de Murcia, **DECLARO**:

Que el Trabajo de Fin de Grado que presento para su exposición y defensa titulado "AN INTRODUCTION TO SUPERSYMMETRY" y cuyos tutores son:

D. JOSE JUAN FERNÁNDEZ MELGAREJO

D. EMILIO TORRENTE LUJÁN

es original y que todas las fuentes utilizadas para su realización han sido debidamente citadas en el mismo.

Murcia, a 14 de JUNIO de 2018.

Firma