Mecánica teórica

pod

Otoño, 2001

Forza generalizada

$$Q_j = \sum \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \; , \quad L(q, \dot{q}, t) = T - V$$

Ec. Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j} = Q_j$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = 0$$

Disipativos

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

$$\mathcal{F} = \frac{1}{2}\sum (k_x v_{x,i}^2 + k_x v_{x,i}^2 + k_x v_{x,i}^2) = \frac{1}{2}\frac{\mathrm{d}W_f}{\mathrm{d}t}$$

Potencial general

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial U}{\partial \dot{q}_j}\right]$$

Acción $S = \int_{t_1}^{t_2} L(q, \dot{q}, y)\mathrm{d}t$, ppi. Hamilton $\delta S = 0$
Momento

$$p_i = \frac{\partial L}{\partial \dot{q}} \; , \qquad p_i = \mathrm{cte} \leftrightarrow L \neq L(q_i)$$

f. energía

$$h(q, \dot{q}, t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \; , \qquad \frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\partial L}{\partial t}$$

Si $L = L_1 + L_2 + L_3$, $\rightarrow V \neq V(\dot{q}) \Rightarrow h = E$

Formulismo de Hamilton

$$H(p, q, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \; , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad , \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

si $L = L_0 + L_1 + L_2$, $\quad L_2 = T \; (q_i \neq q_i(t))$
$\rightarrow \quad L_0 = -V \neq -V(\dot{q})$, $\rightarrow H = L_2 - L_0 = E$

$L = L_0(q, t) + \dot{q}^t \mathbf{a} + \frac{1}{2}\dot{q}^t T \dot{q}$
$H(q, p, t) = \frac{1}{2}(p^t - \mathbf{a}^t)T^{-1}(p - \mathbf{a}) - L_0(q, t)$

$$\delta I = \delta \int_{t_1}^{t_2} L\mathrm{d}t = \delta \int_{t_1}^{t_2} (\dot{q}_i p_i - H)\mathrm{d}t = 0$$

$$\Delta \int_{t_1}^{t_2} L\mathrm{d}t = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} L(\alpha)\mathrm{d}t - \int_{t_1}^{t_2} L(0)\mathrm{d}t$$

Mínima acción

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i \mathrm{d}t = 0$$

Transformaciones canónicas. $\quad K = H + \frac{\partial F}{\partial t}$

$$p_i \dot{q}_i - H(q, p, t) = P_i Q_i - K(Q, P, t) + \frac{\mathrm{d}}{\mathrm{d}t} F(p, P, q, Q, t)$$

1. $F = F_1(q, Q, t)$: $p_i = \frac{\partial F_1}{\partial q_i}$, $\qquad P = -\frac{\partial F_1}{\partial Q_i}$

$$\frac{\partial p_i}{\partial Q_k} = \frac{\partial^2 F_1}{\partial Q_k \partial Q_i} = -\frac{\partial P_k}{\partial q_i}$$

2. $F = F_2(q, P, t) - Q_i P_i$. $p_i = \frac{\partial F_2}{\partial q_i}$, $\qquad Q_i = \frac{\partial F_2}{\partial P_i}$

$$\frac{\partial p_i}{\partial P_k} = \frac{\partial^2 F_2}{\partial P_k \partial q_i} = \frac{\partial Q_k}{\partial q_i}$$

3. $F = q_i p_i + F_3(q, Q, t)$. $q_i = -\frac{\partial F_3}{\partial q_i}$, $\qquad P_i = -\frac{\partial F_3}{\partial Q_i}$

$$\frac{\partial q_i}{\partial Q_k} = -\frac{\partial^2 F_3}{\partial Q_k \partial p_i} = \frac{\partial P_k}{\partial q_i}$$

4. $F = q_i p_i - Q_i P_i + F_4(p,P,t)$. $q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_i = \frac{\partial F_4}{\partial P_i}$

$$\frac{\partial q_i}{\partial P_k} = \frac{\partial^2 F_4}{\partial P_k \partial p_i} = -\frac{\partial Q_k}{\partial q_i}$$

Condiciones

$$\{Q_i, Q_j\} = \{q_i, q_j\} = \{P_i, P_j\} = \{p_i, p_j\} = 0$$

$$\{P_i, Q_j\} = \{p_i, q_j\} = \delta_{ij}$$

Liouville

$$M_{\alpha,\beta} = \frac{\partial x_\alpha}{\partial y_\beta} \ , \ M^t J M = J$$

$$\phi_{t,s}(x) = (\varphi^1_{t,s}(x), \ldots, \varphi^{2n}_{t,s}(x)) \ , \ V_s = V_t$$

Paréntesis de Poisson

$$\{f,g\} = \sum_i^n \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} + \{H,g\} \ , \ \{f\circ\psi, g\circ\psi\} = \{f,g\}\circ\psi$$

Jacobi $\{u,\{v,w\}\} + \{v,\{w,u\}\} + \{w,\{u,v\}\} = 0$
Eq. Hamilton $\dot{q}_k = \{H, q_k\}$, $\dot{p}_k = \{H, p_k\}$

Hamilton-Jacobi $K = H + \frac{\partial F}{\partial t} = 0$, $S = F_2$

$$H\left(q_1,\ldots,q_n;\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n};t\right) + \frac{\partial S}{\partial t} = 0$$

$$p_i = \frac{\partial S}{\partial q_i} = \alpha_i \ , \quad Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S}{\partial \alpha_i} = \beta_i$$

$$S(q,\alpha,t) = W(q,\alpha) - \alpha_1 t \ , \ W(q,\alpha) = \sum_i W_i(q_i,\alpha)$$

si $H \neq H(q_j) \to p_j = P_j = \gamma$, $W_j = \gamma q_1$
Variables acción-angulo $H(q,p) = \alpha_1 = E$
Momento $J = \oint p\mathrm{d}q$

$$Q = \frac{\partial S(q,\alpha,t)}{\partial \alpha_1} \ , \qquad \bar{\omega} = \frac{\partial W(q,J)}{\partial J}$$

$$\dot{\omega} = \frac{\partial H(J)}{\partial J} = \nu(J) \ , \qquad \bar{\omega} = \int \nu(J)\mathrm{d}t = \nu(J)t + \beta$$

S. no inercial

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right)_f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right)_g + \vec{\omega}\times \ , \ \vec{v}_f = \vec{V} + \vec{v}_r + \omega \times \vec{r}$$

$$\vec{F} = m\vec{a}_f = m\ddot{\vec{R}} + m\vec{a}_r + m\vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2m\vec{\omega}\times\vec{v}_r$$

si $\ddot{\vec{R}} = 0 \to \vec{F}_{\mathrm{ef}} = m\vec{a}_r$

$$\vec{F}_{\mathrm{ef}} = m\vec{a}_f - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_r$$

Sólido rígido

$$I_{i,j} = \sum_\alpha m_\alpha \left[\delta_{i,j}\sum_k x^2_{\alpha,k} - x_{\alpha,i}x_{\alpha,i}\right]$$

$$I_{i,j} = \int_v \left[\delta_{i,j}\sum_k x_k^2 - x_i x_j\right]\rho(\vec{r})\mathrm{d}V \ , \ T_{\mathrm{rot}} = \frac{1}{2}\sum_{i,j} I_{i,j}\omega_i\omega_j$$

$$T_{\mathrm{rot}} = \frac{1}{2}\vec{L}\vec{\omega} \ , \ L_i = \sum_j I_{i,j}\ \omega_j \ , \ \vec{L} = \{I\}\ \vec{\omega}$$

$$I^{\mathrm{cm}}_{i,j} = J_{i,j} - M\left[a^2\delta_{i,j} - a_i a_j\right] \ , \ \vec{a} : x_i \to x'^{\mathrm{cm}}_i$$

Angulos de Euler $\lambda = \lambda_\psi\ \lambda_\theta\ \lambda_\varphi$

$$(I_i - I_j)\omega_i\omega_j - \sum_k I_k\dot{\omega}_k\epsilon_{ijk} = 0$$

$$(I_i - I_j)\omega_i\omega_j - \sum_k (I_k\dot{\omega}_k - N_k)\epsilon_{ijk} = 0$$