Ampliación de Mecánica Teórica

pod

Primavera, 2002

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial \vec{q}} = 0 \ , \quad L = T - V$$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}} \ , \quad H = \vec{p}\cdot\dot{\vec{q}} - L$$

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \ , \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{p}} \ , \quad \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, H]$$

$$[f, g] = \frac{\partial f}{\partial \vec{q}}\frac{\partial g}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}}\frac{\partial g}{\partial \vec{q}} = \left(\frac{\partial f}{\partial \vec{x}}\right)^t J \left(\frac{\partial g}{\partial \vec{x}}\right)$$

$$\dot{p}_i = [p_i, H] \ , \quad \dot{q}_i = [q_i, H] \ , \quad [p_i, q_j] = -\delta_{ij}$$

$$\vec{x} = (\vec{q}, \vec{p}) \ , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \dot{\vec{x}} = J\frac{\partial H}{\partial \vec{x}} = [\vec{x}, H]$$

Transformaciones canónicas. $\quad H' = H + \frac{\partial F}{\partial t}$

$$(\vec{p}\ \mathrm{d}\vec{q} - H\mathrm{d}t) - \left(\vec{P}\ \mathrm{d}\vec{Q} - H'\mathrm{d}t\right) = \mathrm{d}F$$

1. $F = F_1(q, Q, t)$:

$$\vec{p} = \frac{\partial F_1}{\partial \vec{q}} \ , \qquad \vec{P} = -\frac{\partial F_1}{\partial \vec{Q}}$$

2. $F = F_2(q, P, t) - \vec{Q}\vec{P}$:

$$\vec{p} = \frac{\partial F_2}{\partial \vec{q}} \ , \qquad \vec{Q} = \frac{\partial F_2}{\partial \vec{P}}$$

3. $F = \vec{q}\vec{p} + F_3(q, Q, t)$:

$$\vec{q} = -\frac{\partial F_3}{\partial \vec{q}} \ , \qquad \vec{P} = -\frac{\partial F_3}{\partial \vec{Q}}$$

4. $F = \vec{q}\vec{p} - \vec{Q}\vec{P} + F_4(p, P, t)$:

$$\vec{q} = -\frac{\partial F_4}{\partial \vec{p}} \ , \quad \vec{Q} = \frac{\partial F_4}{\partial \vec{P}}$$

$$\dot{\vec{X}} = MJM^t\frac{\partial H}{\partial \vec{X}} \to MJM^t = J$$

Invariante integral $\frac{\mathrm{d}}{\mathrm{d}t}I = 0$

$$I(t) = \int_{\Omega_t}\cdots\int \mathrm{d}\vec{p}\mathrm{d}\vec{q}\ \rho(\vec{q}, \vec{p}, t) \leftrightarrow \frac{\partial \rho}{\partial t} + [\rho, H] = 0$$

$$\frac{\partial S}{\partial t} + H\left(\vec{q}, \frac{\partial S}{\partial \vec{q}}, t\right) = 0 \quad , \quad \vec{p} = \frac{\partial S}{\partial \vec{q}} \ , \quad \vec{\beta} = \frac{\partial S}{\partial \vec{\alpha}}$$

$$J_k(\vec{\alpha}) = \oint p_k(q_k, \vec{\alpha})\mathrm{d}q_k \quad , \quad \dot{\vec{\omega}} = \frac{\partial H'}{\partial \vec{J}} = \vec{\nu}(\vec{J})$$

Variació nde las constantes $H = H_0 + \lambda H_1$, t.c S_0

$$\begin{aligned} \vec{\alpha}_1(t) &= -\int \mathrm{d}t\frac{\partial H_1}{\partial \vec{\beta}} \\ \vec{\alpha}_2(t) &= -\int \mathrm{d}t\left\{\frac{\partial^2 H_1}{\partial \vec{\alpha}\ \partial \vec{\beta}}\cdot\vec{\alpha}_1 + \frac{\partial^2 H_1}{\partial \vec{\beta}\ \partial \vec{\beta}}\cdot\vec{\beta}_1\right\} \\ \vec{\beta}_1(t) &= \int \mathrm{d}t\frac{\partial H_1}{\partial \vec{\alpha}} \\ \vec{\beta}_2(t) &= \int \mathrm{d}t\left\{\frac{\partial^2 H_1}{\partial \vec{\beta}\ \partial \vec{\alpha}}\cdot\vec{\beta}_1 + \frac{\partial^2 H_1}{\partial \vec{\alpha}\ \partial \vec{\alpha}}\cdot\vec{\alpha}_1\right\} \end{aligned}$$

Met.media

$$H'(\alpha, \beta, t) \approx \langle H_1(\alpha, \beta, t)\rangle = \frac{1}{T_0}\int_{T_0}\mathrm{d}t'\ H_1(\alpha, \beta, t')$$

Poincaré-Von Ziepel

$$E_1 = H_{\vec{0}}^{(1)} = \int_{[0,1]^n}\mathrm{d}\vec{\omega}_0 H_1(\vec{J}) \equiv \left\langle H_1(\vec{J})\right\rangle$$

$$E_2 = \left\langle H_2(\vec{J}, \vec{\omega}_0)\right\rangle + \frac{1}{2}\left\langle\frac{\partial S_1}{\partial \vec{\omega}_0}\cdot\frac{\partial^2 H_0}{\partial \vec{J}\ \partial \vec{J}}\cdot\frac{\partial S_1}{\partial \vec{\omega}_0}\right\rangle + \left\langle\frac{\partial H_1}{\partial \vec{J}}\cdot\frac{\partial S_1}{\partial \vec{\omega}_0}\right\rangle$$

$$S_1(\vec{\omega}_0, \vec{J}) = \frac{-1}{2\pi i} \sum_{\vec{m}\neq 0} \frac{H^{(1)}_{\vec{m}}(\vec{J})}{\vec{m}\cdot\vec{\nu}_0(\vec{J})}\, e^{2\pi i \vec{m}\cdot\vec{\omega}_0}$$

$$\vec{\nu} = \vec{\nu}_0 + \epsilon\, \partial[E_1]\vec{J} + \epsilon^2\, \partial[E_2]\vec{J} + \cdots$$

$$J_0 = J + \epsilon \frac{\partial S_1}{\partial \vec{\omega}_0} + \cdots$$

$$\vec{\omega} = \omega_0 + \epsilon \frac{\partial S_1}{\partial \vec{J}} + \cdots$$

$$(n=1)\ E_1(J) = H_1(\mathrm{J}, \omega_0) + \nu_0(\vec{J})\frac{\partial S_1}{\partial \omega_0}$$

- Taylor $E = H_0(\frac{\partial S}{\partial \vec{\omega}_0}) + \epsilon H_1(\frac{\partial S}{\partial \vec{\omega}_0}, \vec{\omega}_0) + \cdots$
- Desenvol i $E = H_0' + \epsilon H_1' + \cdots$
- Fourier $S_{\vec{m}}$
- Integrar $\vec{\omega}_0$.

Twist Map

$$J_{n+1} = J_n \quad , \quad \omega_{n+1} = \alpha(J_{n+1}) + \omega_n$$

Pertorb. ϵH_1

$$J_{n+1} = J_n + \epsilon f(J_{n+1}, \omega_n)$$

$$\omega_{n+1} = \alpha(J_{n+1}) + \omega_n + \epsilon g(J_{n+1}, \omega_n)$$

Teor. adiabática

$$\lambda = \lambda(\epsilon t) \Longrightarrow \Delta J = o(\epsilon^2)$$

Sist. dinámicos $\dot{\vec{x}} = F(\vec{x}, t)$

$$\vec{k} = \text{cte}\ ,\ \vec{F} = \vec{F}(\vec{x}) \Longleftrightarrow \vec{F}\cdot\overrightarrow{\text{grad}} = 0$$

$$I(t) = \int_{\Omega(t)} \cdots \int \mathrm{d}^N\vec{x}\ \rho(\vec{x}, t) = \text{cte} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\partial \rho}{\partial t} + \text{div}\left[\rho(\vec{x}, t)\vec{F}(\vec{x}, t)\right] = 0$$

Rectificat $\{x_i\} \to \{y_i\}$ tq $y_i = \delta_{1i}$

$$F_k(\vec{x}) = \sum_l \left.\frac{\partial F_k}{\partial x_l}\right|_{\vec{x}=\vec{0}} x_l + o(\vec{x}^2) = \sum_l a_{kl}\, x_l\ , \quad \dot{\vec{x}} = A\vec{x}$$

$$\lambda_k = \alpha_k + i\omega_k \longrightarrow x_k = \sum_l c_{kl} e^{\alpha_l t}\cos(\omega_l t + \varphi_l)$$

- $\lambda_1 \neq \lambda_2 \in \mathbb{R} - \{0\}$
  - $\lambda_1/\lambda_2 > 0$ nodo, parábola ($\lambda_i > 0$ inestable)
  - $\lambda_1/\lambda_2 < 0$ p. ensilladura, hiperbola
- $\lambda_1 = \lambda_2^*$ foco ($\alpha > 0$ inestable, $\alpha = 0$ centro)
- $\lambda_1 = \lambda_2$ nodo.

Oscil. amónic (frec. ϖ)

$$q = \sqrt{\frac{J_0}{\pi\varpi m}} \sin 2\pi\omega_0\ , \quad p = \sqrt{\frac{J_0\varpi m}{\pi}} \cos 2\pi\omega_0$$

$$F = \sum_{\vec{\ell}} a_{\vec{\ell}}\, e^{2\pi i \vec{\ell}\vec{\omega} t} \text{ on } \quad a_{\vec{\ell}} = e^{2\pi i \vec{\beta}\vec{\ell}} \int_{[0,1]^N} \mathrm{d}^n\vec{\omega}\ F\ e^{-2\pi i \vec{\ell}\vec{\omega}}$$