Mecánica: Movimiento unidimensional y osciladores

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1. MOVIMIENTO UNIDIMENSIONAL

$$\boxed{m\ddot{x} = \sum_{i=1}^{N} F_i(x, \dot{x}, t)}$$

Teorema del momento lineal i teorema de la energía:

$$I \equiv p_2 - p_1 = \int_{t_1}^{t_2} F(x, \dot{x}, t)\mathrm{d}t$$

$$\Delta T \equiv T_2 - T_1 = \int_{x_1}^{x_2} F(x, \dot{x}, t)\mathrm{d}x$$

Fuerza dependiente de la velocidad: $F = F(v)$

$$\int_{v_0}^{v} \frac{\mathrm{d}v'}{F(v')} = \int_0^t \frac{\mathrm{d}t'}{m} \rightarrow v(t) = \Phi^{-1}\left(\frac{t}{m}, v_0\right)$$

$$x(t) = x_0 + \int_0^t \Phi^{-1}\left(\frac{t'}{m}, v_0\right)\mathrm{d}t'$$

Fuerza dependiente del tiempo (aplicación del teorema del momento lineal)

$$v(t) = v_0 + \frac{1}{m}\int_0^t F(t')\mathrm{d}t'$$

$$x(t) = x_0 + v_0 t + \frac{1}{m}\int_0^t \left[\int_0^{t'} F(t'')\mathrm{d}t''\right]\mathrm{d}t'$$

Sistemas conservativos: $F = F(x)$

Energía potencial:

$$U(x) = \int_x^{x_{\text{ref}}} F(x')\mathrm{d}x' = -\int_{x_{\text{ref}}}^{x} F(x')\mathrm{d}x' \quad \rightarrow \quad F(x) = -\frac{\mathrm{d}U}{\mathrm{d}x}$$

Conservación de la energía:

$$E = T + U = \frac{1}{2}mv^2 + U(x) = \mathcal{C} \Rightarrow \Delta T = \Delta U$$

Puntos de retorno (x_r): $E = U(x_r) \rightarrow v(x_r) = 0$

Puntos de equilibrio (x_e):

$$\left.\frac{\mathrm{d}U}{\mathrm{d}x}\right|_{x_e} = 0 \begin{cases} U''(x) > 0 \rightarrow \text{estable.} \\ U''(x) < 0 \rightarrow \text{inestable} \end{cases}$$

x(t):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{\frac{2[E - U(x)]}{m}}$$

Oscilador armónico simple:

$$U(x) = \frac{1}{2}kx^2 \quad \rightarrow \quad F(x) = -kx$$

Pequeñas oscilaciones:

$$U(x) = U(x_e) + \underbrace{U'(x)_{x_e}}_{=0}(x - x_e) + \frac{1}{2}\underbrace{U''(x)_{x_e}}_{\equiv k}\underbrace{(x - x_e)}_{\rightarrow 0}{}^2 + \mathcal{O}(3)$$

$$x(t) = A\sin(\omega t + \theta) \rightarrow \begin{cases} A = \sqrt{x_0^2 + \dfrac{v_0^2}{\omega^2}} \\ \theta = \arctan\left(\dfrac{x_0\omega}{v_0}\right) \end{cases}$$

2. OSCILADORES ARMÓNICOS AMORTIGUADOS Y FORZADOS

Oscilador amortiguado:

$$m\ddot{x} + b\dot{x} + kx = 0 \quad \rightarrow \quad \left[\begin{array}{l} x = e^{pt} \\ x = te^{pt}(\Delta = 0) \end{array}\right.$$

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \qquad \left[\begin{array}{l} \gamma \equiv \dfrac{b}{2m} \\ \omega_0^2 \equiv \dfrac{k}{m} \end{array}\right.$$

Sobreamortiguado: $\gamma^2 > \omega_0^2$

$$\boxed{x(t) = C_1\mathrm{e}^{-\gamma_1 t} + C_2\mathrm{e}^{-\gamma_2 t}}$$

$$x(t) = \frac{(\gamma_1 x_0 + v_0)\mathrm{e}^{-\gamma_2 t} - (\gamma_2 x_0 + v_0)\mathrm{e}^{-\gamma_1 t}}{\gamma_1 - \gamma_2}$$

Infraamortiguado: $\gamma^2 < \omega_0^2$

$$\boxed{x(t) = A\mathrm{e}^{-\gamma t}\cos(\omega_1 t + \theta)}$$

$$x(t) = x_0\mathrm{e}^{-\gamma t}\frac{\cos(\omega_1 t + \theta)}{\cos\theta} \quad \rightarrow \quad \left[\begin{array}{l} \omega_1 = \sqrt{\omega_0^2 - \gamma^2} < \omega_0 \\ \theta = \arctan\left(-\dfrac{\gamma x_0 + v_0}{\omega_1 x_0}\right) \end{array}\right.$$

Crítico: $\gamma^2 = \omega_0^2$

$$\boxed{x(t) = C_1\mathrm{e}^{-\gamma t} + C_2 t\mathrm{e}^{-\gamma t}}$$

$$x(t) = x_0\mathrm{e}^{-\gamma t} + (v_0 + \gamma x_0)t\mathrm{e}^{-\gamma t}$$

Oscilador forzado:

$$m\ddot{x} = -kx + b\dot{x} + F(t); \qquad F(t) = F_0\cos(\omega t)$$

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}\sin(\omega t + \beta)$$

$$\beta = \arctan\left(\frac{\omega_0^2 - \omega^2}{2\gamma\omega}\right)$$

Resonancia

$$D(\omega)=\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2-(2\gamma\omega)^2}}$$

Resonancia en amplitud: Oscilador infraamortiguado.

$$\omega_r=\sqrt{\omega_0^2-2\gamma^2}\qquad D(\omega_r)=\frac{F_0/m}{\sqrt{(\omega_0^2-\omega_r^2)^2-(2\gamma\omega_r)^2}}$$

$$D(\omega_r)=\frac{F_0/m}{\sqrt{4\gamma^2(\omega_0^2-\gamma^2)}}$$

Resonancia en potencia:

$$P=F\cdot v\qquad \langle P(t)\rangle=\frac{1}{T}\int\limits_0^T P(t)\mathrm{d}t$$

$$\bar{P}(\omega)=\frac{F_0^2}{4\gamma m}\cdot\frac{1}{\left(\frac{\omega_0^2-\omega^2}{2\gamma\omega}\right)^2+1}\qquad \bar{P}(\omega_0)=\frac{F_0^2}{4\gamma\omega}$$

Osciladores Acoplados

$$\begin{cases} m\ddot{x}_1=-kx_1+k_3(x_2-x_1)\\ m\ddot{x}_2=-kx_2+k_3(x_1-x_2)\end{cases}\begin{Bmatrix} y\equiv x_1+x_2\\ z\equiv x_2-x_1\end{Bmatrix}\rightarrow\begin{Bmatrix} m\ddot{y}=-ky\\ m\ddot{z}=-(k-2k_3)z\end{Bmatrix}$$

Solución:

$$x_1(t)=A_0\cos(\omega_0 t+\phi)-A_1\cos(\omega_1 t+\psi)\qquad A_0=A_y/2;\ A_1=A_z/2$$

$$x_2(t)=A_0\cos(\omega_0 t+\phi)+A_1\cos(\omega_1 t+\psi)\quad \omega_0=\sqrt{\frac{k}{m}};\ \omega_1=\sqrt{\omega_0+\frac{2k_3}{m}}$$

Oscilaciones concordantes: $x_1(0)=x_2(0)=x_0,\ \ \dot{x}_i(0)=0$

$$\begin{cases} x_1(t)=x_0\cos(\omega_0 t)\\ x_2(t)=x_0\cos(\omega_0 t)\end{cases}$$

Osc. contrapuestas: $x_1(0)=x_0,\ -x_2(0)=-x_0,\ \ \dot{x}_i(0)=0$

$$\begin{cases} x_1(t)=x_0\cos(\omega_1 t)\\ x_2(t)=-x_0\cos(\omega_1 t)\end{cases}$$

Oscilaciones pulsantes: $x_1(0)=x_0,\ x_2(0)=0,\ \ \dot{x}_i(0)=0$

$$\begin{cases} x_1(t)=x_0\cos(\Omega_- t)\cos(\Omega_+ t)\\ x_2(t)=x_0\sin(\Omega_- t)\sin(\Omega_+ t)\end{cases}$$

$$\Omega_+\equiv\frac{\omega_0+\omega_1}{2};\qquad \Omega_-\equiv\frac{\omega_1-\omega_0}{2}$$

Solución mediante diagonalización de matrices

$$\ddot{X}=AX;\ddot{X}=\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix};\ A=\begin{pmatrix}a_{11}\ a_{12}\\ a_{21}\ a_{22}\end{pmatrix};\ X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=\begin{pmatrix}a_{11}\ a_{12}\\ a_{21}\ a_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\mapsto\begin{pmatrix}\ddot{y}_1\\ \ddot{y}_2\end{pmatrix}=\begin{pmatrix}-\omega_1^2 & 0\\ 0 & -\omega_2^2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$$\det(A-\lambda I)=0\mapsto\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\mapsto -\omega_i^2=\lambda_i$$

Para relacionar X con las Y: matriz del cambio de base.

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}a\ c\\ b\ d\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}\qquad \vec{v}_1=\begin{pmatrix}a\\ b\end{pmatrix}\qquad \vec{v}_2=\begin{pmatrix}c\\ d\end{pmatrix}$$

$$\begin{cases} A\vec{v}_1=\lambda_1\vec{v}_1\\ A\vec{v}_2=\lambda_2\vec{v}_2\end{cases}$$